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Mathematical Properties of Position-Space Renormalization-Group Transformations

Robert B. Griffiths[1] and Paul A. Pearce[1]

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Properties of "position-space" or "cell-type" renormalization-group transformations from an Ising model object system onto an Ising model image system, of the type introduced by Niemeijer, van Leeuwen, and Kadanoff, are studied in the thermodynamic limit of an infinite lattice. In the case of a Kadanoff transformation with finite p, we prove that if the magnetic field in the object system is sufficiently large (i.e., the lattice-gas activity is sufficiently small), the transformation leads to a well-defined set of image interactions with finite norm, in the thermodynamic limit, and these interactions are analytic functions of the object interactions. Under the same conditions the image interactions decay exponentially rapidly with the geometrical size of the clusters with which they are associated if the object interactions are suitably short-ranged. We also present compelling evidence (not, however, a completely rigorous proof) that under other conditions both the finite- and infinite-p ("majority rule") transformations exhibit peculiarities, suggesting either that the image interactions are undefined (i.e., the transformation does not possess a thermodynamic limit) or that they fail to be smooth functions of the object interactions. These peculiarities are associated (in terms of their mathematical origin) with phase transitions in the object system governed not by the object interactions themselves, but by a modified set of interactions.

KEY WORDS: Renormalization group; Ising; phase transition.

1. INTRODUCTION

Renormalization-group methods have played an extremely important role in statistical mechanics in the last few years.(1) They have been remarkably successful when applied to a variety of problems in the field of phase transitions and critical phenomena. Almost all the interesting applications have

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[1] Physics Department, Carnegie-Mellon University, Pittsburgh, Pennsylvania.

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involved transformations which are only approximate. Usually it is assumed that these are approximations to transformations which, in principle at least, are exact, but are too difficult to carry out in practice. Despite the obvious practical importance of renormalization-group methods, very little seems to be known about the mathematical properties[2] of exact transformations except in the few cases where they can be carried out explicitly in closed form.[3] In this paper we are concerned with the question of whether such exact transformations actually exist in the thermodynamic limit, and whether they have certain properties which are commonly assumed (implicitly if not explicitly) in actual applications.

A wide variety of renormalization-group transformations have been used to study phase transitions. In this paper we shall consider "position space" or "real space" (in contrast to "momentum space") or "cell" types of transformations of Ising models onto Ising models.[4–6] These transformations can be described fairly simply, so there is no particular difficulty in posing precise mathematical questions. In addition, a great deal is known about the existence or nonexistence of phase transitions and the properties of correlation functions in Ising models, and some of this information is quite useful in working out properties of the transformations.

A renormalization-group transformation is usually described as a mapping $\mathscr{R}$ from one Hamiltonian or set of interactions (we use the two terms interchangeably) onto another,

$$H' = \mathscr{R}(H)$$

In applications it is usually assumed that $\mathscr{R}$ has the following properties in the thermodynamic limit of an infinite system:

(i) The transformation $\mathscr{R}$ is well defined for H belonging to a suitable class of interactions.

(ii) If the interactions in H decrease rapidly with distance (in some appropriate sense), those in $\mathscr{R}(H)$ also decrease rapidly with distance (in the same or perhaps some different sense).

(iii) The transformation $\mathscr{R}$ is smooth in the sense that the various terms in $\mathscr{R}(H)$ depend smoothly on parameters which appear in H.

(iv) The transformation $\mathscr{R}$ possesses various fixed points.

In this paper we shall be concerned solely with properties (i)–(iii). In most applications of renormalization-group methods, these properties are exhibited explicitly by the *approximate* transformations employed, and since the thermodynamic limit is taken implicitly rather than explicitly, the issues raised by such a limit are not discussed. Nonetheless, it is our opinion—

[2] Probability distributions for block spins have been studied by Gallavotti and coworkers.[2]

[3] A number of examples are discussed by Nelson and Fisher.[3]

amply confirmed, we believe, by the results reported below—that the existence of properties (i)–(iii) for an *exact* $\mathscr{R}$ is a nontrivial matter. Our results are both positive and negative. In certain cases corresponding to low density or high magnetic field (in lattice-gas and magnetic language, respectively), we are able to show quite rigorously that at least some interesting types of transformations do possess properties (i)–(iii). In other cases we are able to show, or at least present very plausible arguments, that the transformations do not possess at least one of these properties and that they instead exhibit a rather peculiar behavior. The peculiarities are closely related to phase transitions in a system whose Hamiltonian is a modified form of H. So far as we know, such peculiarities have not been seen in approximate transformations, or at least their source has not been identified as due to a modified Hamiltonian.

Position-space (and other) renormalization-group transformations are customarily introduced by means of formulas which make perfectly good sense for a finite system, and we follow this procedure in Section 2.1. The task of extending these transformations so that they possess a proper thermodynamic limit has received little if any attention in the renormalization-group literature, and hence we provide in Section 2.2 an informal introduction to and motivation for the formal procedures described in Sections 3 and 4: states of infinite systems and equilibrium equations, respectively. These procedures are used in Section 5, together with the equations of Gallavotti and Miracle-Sole, to establish properties (i)–(iii) for Kadanoff transformations with finite p in the limit of low activity. Various examples in which at least one of these properties seems to be violated are discussed in Section 6. Our conclusions are summarized in Section 7 along with a discussion of their possible significance for the renormalization-group enterprise.

2. POSITION-SPACE RENORMALIZATION-GROUP TRANSFORMATIONS

2.1. Transformation on Finite Systems

The general transformation we wish to consider relates the "Hamiltonian" or set of interactions H' for an "image" system to the interactions H for a finite "object" system through a formula

$$\exp H'(\tau) = \mathrm{Tr}_\sigma[T(\tau, \sigma) \exp H(\sigma)] \tag{2.1}$$

Here σ stands for a collection of Ising spin variables $\sigma_i = \pm 1$, which form the object system, where the subscript i labels sites of a finite lattice Ω. The real-valued function $H(\sigma)$ is the usual Hamiltonian multiplified by a factor of $(-1/kT)$. The Ising variables $\tau_i = \pm 1$ for i in a finite lattice Ω', denoted collectively by τ, form the image system. The quantity $T(\tau, \sigma)$, which may be regarded as a conditional probability, satisfies the conditions

$$T(\tau, \sigma) \geqslant 0 \tag{2.2}$$

$$\mathrm{Tr}_\tau[T(\tau, \sigma)] = \sum_{\tau_1} \sum_{\tau_2} \cdots T(\tau, \sigma) = 1 \tag{2.3}$$

We use the abbreviation Tr_τ (trace) for a multiple sum over all the τ variables, and Tr_σ for the corresponding sum over the σ variables. The condition (2.2) ensures that the right side of (2.1) is nonnegative, and hence H' is well defined (though it might possibly have the value $-\infty$).

The Gibbs probability distributions ρ and ρ' for the object and image systems, respectively, are given by the usual formulas:

$$\rho(\sigma) = [\exp H(\sigma)]/\mathrm{Tr}_\sigma[\exp H(\sigma)] \tag{2.4}$$

$$\rho'(\tau) = [\exp H'(\tau)]/\mathrm{Tr}_\tau[\exp H'(\tau)] \tag{2.5}$$

From (2.1) and (2.3) it is evident that

$$\rho'(\tau) = \mathrm{Tr}_\sigma[T(\tau, \sigma)\rho(\sigma)] \tag{2.6}$$

Hence it is possible to think of the transformation from H to H' in (2.1) as consisting of three steps indicated symbolically by

$$H \to \rho \to \rho' \to H' \tag{2.7}$$

where the first two arrows correspond to formulas (2.4) and (2.6), while the third is the inverse of (2.5): finding a set of interactions H' which will generate ρ'. [Strictly speaking, (2.5) will only define H' up to an additive constant which is unambiguously determined by (2.1), but this is not important for our purposes.] Although (2.1) and (2.7) are equivalent for finite systems, (2.7) has some advantages when discussing the thermodynamic limit, as we shall see in Section 2.2.

In Sections 5 and 6, we shall be interested in transformations T of the following form. We suppose there is a function mapping Ω, or a subset of Ω, onto Ω' with the significance that the set $C(j)$ of sites in Ω which are mapped onto a specific site j in Ω' are the object spins lying within the jth "cell" and thus associated with the image spin τ_j. The Kadanoff[5,6] transformation is given by

$$T(\tau, \sigma) = \prod_{j\in\Omega'} \left[2\cosh\left(p \sum_{i\in C(j)} \sigma_i\right)\right]^{-1} \exp\left[p\tau_j \sum_{i\in C(j)} \sigma_i\right] \tag{2.8}$$

where p is a real number which may be thought of as a coupling between the object spins in a particular cell and the corresponding image spin. One could, of course, allow p to vary from cell to cell, but we shall only consider the case in which it is a constant, and in which all the cells have the same size and shape and form a regularly spaced lattice.

We shall refer to the special case in which each site in Ω is in a separate cell as model I. In this case (2.8) has the simple form

$$T(\tau, \sigma) = (2\cosh p)^{-|\Omega|} \exp\left[p \sum_{j\in\Omega} \tau_j \sigma_j\right] \tag{2.9}$$

where we denote by $|A|$ the number of elements in a set A, and where the corresponding sites in Ω' and Ω are labeled with the same index j. A generalization of model I, called model II, is obtained by using cells $C(j)$ containing only one object site, which we denote by (j), but in which the union of the C_j comprises only some fraction of the sites in Ω. That is, each image spin is coupled to a single object spin, but there are object spins coupled to no image spin. The appropriate transformation formula,

$$T(\tau, \sigma) = (2\cosh p)^{-|\Omega'|} \exp\left[p \sum_{j\in\Omega'} \tau_j \sigma_{(j)}\right] \tag{2.10}$$

is similar to (2.9). Note that the special sites (j) in Ω are assumed to form a regular lattice. Neither model I nor model II is of practical interest for renormalization-group calculations, but their simple structure makes them useful for studying the thermodynamic limit.

The transformations (2.8)–(2.10) have well-defined limits as p goes to $+\infty$. In particular, (2.8) becomes the "majority rule" transformation first introduced by Niemeijer and van Leeuwen,[(4)] (2.10) becomes a "decimation" transformation,[4] and (2.9) becomes the identity transformation in this limit.

2.2. The Thermodynamic Limit

The problems which arise when taking the thermodynamic limit of (2.1) or (2.7) are best discussed in terms of a specific example. Let Ω be all the sites of a square (or simple cubic) lattice falling inside a large square (or cube), and let

$$H(\sigma) = K \sum_{\langle ij\rangle} \sigma_i \sigma_j + h \sum_i \sigma_i \tag{2.11}$$

where $\langle ij\rangle$ denotes a nearest-neighbor pair of sites and the summations are restricted to sites within Ω. Let T be of the form (2.8). Let the Gibbs distributions for the object and image system and the Hamiltonian for the image system be $\rho_\Omega(\sigma)$, $\rho_\Omega'(\tau)$, and $H_\Omega'(\tau)$, respectively. We now wish to know what happens to each of these quantities as Ω expands until it encompasses all the sites of an infinite lattice $\mathscr{L}$, and Ω' (which we assume is also a large square or cube) expands in a corresponding manner.

[4] The name comes about because ρ' can be calculated by summing $\rho(\sigma)$ over all σ_j *except* at the special sites (j), and regarding the result as a function of the remaining variables $\sigma_{(j)} = \tau_j$. The variables summed over are considered "decimated."

While it is not useful to speak of the probability of a particular configuration of an infinite system, one can introduce a probability measure ρ (the symbol μ is used from Section 3 on for the lattice-gas counterpart of ρ) on sets of configurations, and there is a well-defined sense in which ρ_Ω can converge to a limit ρ. In general this limit is not unique, though one can always choose a *particular* sequence of squares Ω such that ρ_Ω converges to a limit. For the transformations T of interest to us, convergence of the ρ_Ω guarantees that of the ρ_Ω', and the limits are related by the analog of (2.6) for an infinite system. (Details are given in Section 3.)

The image Hamiltonian $H_\Omega'(\tau)$ will always exist if Ω is finite, but its form will, in general, be much more complicated than (2.11): there will be one-spin interactions, two-spin interactions (not limited to nearest neighbors), three-spin, etc., up to and including a term involving all the τ's in Ω'. And unlike (2.11), where Ω appears only in the limits of summation, one must expect that *each* term in H_Ω' will have an *explicit* dependence on Ω. For example, H_Ω' will include a term $K_{ij}'\tau_i\tau_j$ for a particular pair of sites i and j, and K_{ij}' will be a function of Ω as well as of the K and h in (2.11).

The first problem in showing the existence of a thermodynamic limit for H' is to demonstrate that K_{ij}', along with all the other coefficients in H', tends to a well-defined limit as Ω becomes infinite. *In addition* one must show that the limiting H' has a proper relationship with the limiting probability measure ρ'. This requirement is not trivial, as can be seen by considering the following example. For a finite system Ω', let

$$H_\Omega' = K'|\Omega'|^{-1} \sum_{i \neq j} \tau_i\tau_j + h \sum_i \tau_i \tag{2.12}$$

This is the well-known "equivalent-neighbor" model, which gives rise to a "mean-field" phase transition as $|\Omega'|$ becomes infinite (see, e.g., Ref. 7). Note, however, that the interaction between any pair of spins tends to zero as $|\Omega'| \to \infty$. Thus one might be tempted to conclude that the thermodynamic limit for H' is independent of the value of K'. The limiting probability distribution, however, depends on K'.

This example is artificial in that (2.12) does not arise as the result of a transformation of the form (2.1). However, there is no reason to suppose a priori that such transformations do not yield equally "pathological" results, and hence one needs to require something stronger than term-by-term convergence of the interactions in H_Ω' in order to achieve a satisfactory thermodynamic limit for (2.1).

An alternative approach is to try and work out the properties of a transformation of the form (2.7) applied to an infinite system. The first arrow in (2.7), associating the state of an infinite system with a set of interactions, has been studied extensively.[8–11] In particular, the equilibrium equations introduced by Dobrushin[9] and by Lanford and Ruelle[10,11] permit one to associate with any H satisfying some fairly mild conditions a "Gibbs state" ρ, and hence are a generalization of (2.4). (There may be more than one ρ associated with a particular H.) The second arrow in (2.7), from ρ to ρ', is easy to define for an infinite system (Section 3.3). Finally, if ρ' is a Gibbs state corresponding to some H', the latter is unique.[12] Thus the final step in (2.7) (though not the first step) is unambiguous within this framework, if it can be carried out at all.

In this paper we adopt the point of view that the thermodynamic limit of a renormalization-group transformation on an Ising model consists of the three steps in (2.7) carried out for an infinite system. However, our arguments are based in large part on taking the infinite limit of a finite system. This is useful both as a source of physical insight and as a method for constructing proofs. Thus, from another point of view, our arguments may be considered as supplying the technical details of sufficient conditions to rule out pathological behavior similar to that of (2.12).

The concepts of states of an infinite system and equilibrium equations for these states are, unfortunately, somewhat technical and not familiar to many physicists working in statistical mechanics. Since they are an essential part of our arguments in Sections 5 and 6, and also because we require certain (mostly minor) generalizations of previously published results, the necessary facts are presented as a series of theorems in Sections 3 and 4. The proofs (in some cases) and literature references (in other cases) will be found in Appendix A.

3. STATES OF AN INFINITE SYSTEM

3.1. Continuous Functions

In this section we shall use "lattice-gas" rather than "Ising-spin" language. The connection is as follows. Let $\mathscr{L}$ be a countably infinite set of points or lattice sites. If $\sigma_j = +1$ we shall say that the site j is occupied by a particle, while if $\sigma_j = -1$ the site is empty. A configuration X is the set of occupied sites in $\mathscr{L}$, and the space of configurations is thus the set

$$\mathscr{K} = \mathscr{P}(\mathscr{L}) \tag{3.1}$$

of all subsets of $\mathscr{L}$. In $\mathscr{K}$ we introduce a topology with a base given by the cylinder sets

$$I_\Lambda(A) = \{X \in \mathscr{K}: \quad X \cap \Lambda = A\} \tag{3.2}$$

where Λ is a finite subset of $\mathscr{L}$ and A is a subset of Λ. We shall use the letters A, B, C, Λ, and Ω to denote *finite* subsets of $\mathscr{L}$ (including the null subset $\varnothing$), and X, Y, and Z for subsets which may be finite or infinite.

The topology of $\mathscr{K}$ can be conveniently characterized by sequences. A sequence X_j of subsets of $\mathscr{L}$ converges to a set X if for every finite Λ,

$$X_j \cap \Lambda = X \cap \Lambda \tag{3.3}$$

for all j sufficiently large. The space $\mathscr{K}$ is compact, which means that every sequence X_j possesses some subsequence which converges to some X. A real or complex-valued function f on $\mathscr{K}$ is *continuous* if

$$f(X) = \lim_{j\to\infty} f(X_j) \tag{3.4}$$

for every X and every sequence X_j converging to X. It is uniformly continuous if, given any $\epsilon > 0$, there is a finite Λ such that

$$|f(X) - f(Y)| < \epsilon \tag{3.5}$$

whenever

$$X \cap \Lambda = Y \cap \Lambda \tag{3.6}$$

A sequence of functions f_j converges uniformly to f if for any $\epsilon > 0$ there is a k such that $j \geqslant k$ implies that

$$|f(X) - f_j(X)| < \epsilon \tag{3.7}$$

independent of X.

Theorem 3.1. Any continuous function f on $\mathscr{K}$ is uniformly continuous, bounded, and uniquely determined by its values on the finite subsets of $\mathscr{L}$. If a sequence of continuous functions f_j converges uniformly to a function f, then f is continuous.

Theorem 3.2. Let f be a function defined on the finite subsets of $\mathscr{L}$ with the property that for any $\epsilon > 0$ there is a finite Λ such that (3.5) is satisfied whenever (3.6) holds for finite sets X and Y. Then there is a unique extension of f to a continuous function on $\mathscr{K}$.

3.2 Measures and Integration

The appropriate generalization of a probability distribution for a finite system Ω is a measure on $\mathscr{K}$. The measures of interest to us are completely characterized by their values on cylinder sets of the form (3.2).

Theorem 3.3. Suppose that for every finite $\Lambda \subset \mathscr{L}$ and every $A \subset \Lambda$, $\mu_\Lambda(A)$ is a real, positive number:

$$0 \leqslant \mu_\Lambda(A) < \infty \tag{3.8}$$

In addition, suppose that these numbers satisfy a consistency condition

$$\mu_M(B) = \sum_{A\subset\Lambda: A\cap M = B} \mu_\Lambda(A) \tag{3.9}$$

for every finite Λ, every $M \subset \Lambda$, and every $B \subset M$. Then there is a unique measure $\mu(dY)$ on the σ-ring generated by the cylinder sets $I_\Lambda(A)$ with the property that

$$\mu(I_\Lambda(A)) = \int_{I_\Lambda(A)} \mu(dY) = \mu_\Lambda(A) \tag{3.10}$$

Furthermore, if f is a continuous function on $\mathscr{K}$, its integral with respect to this measure is given by

$$\int f(Y)\,\mu(dY) = \lim_{\Lambda\to\mathscr{L}} \sum_{A\subset\Lambda} f(A)\mu_\Lambda(A) \tag{3.11}$$

Henceforth in this paper the term "measure" will always refer to a measure of the type considered in the previous theorem. The *state* of an infinite system is a *probability* measure, one for which

$$\mu(\mathscr{K}) = 1 \tag{3.12}$$

or, equivalently, the linear functional on continuous functions given by (3.11) for such a measure.

A sequence of measures μ_j will be said to converge to a measure μ if for every finite Λ and $A \subset \Lambda$,

$$\mu(I_\Lambda(A)) = \lim_{j\to\infty} \mu_j(I_\Lambda(A)) \tag{3.13}$$

In fact, if for each Λ and $A \subset \Lambda$ the limit on the right side exists as a finite number, which we can designate by $\mu_\Lambda(A)$, it is easy to check that these numbers satisfy (3.8) and (3.9) and hence generate a measure μ by Theorem 3.3.

Theorem 3.4. Let f_j be a sequence of continuous functions converging uniformly to a function f, and μ_j a sequence of measures converging to μ; then

$$\int f(Y)\,\mu(dY) = \lim_{j\to\infty} \int f_j(Y)\,\mu_j(dY) \tag{3.14}$$

3.3. Renormalization Transformations

Let $\mathscr{L}$ and $\mathscr{L}'$ be the countably infinite lattices for the object and image system, respectively, in the thermodynamic limit. A *continuous conditional probability* $T(dX|Y)$ has the following properties: (i) For a fixed $Y \subset \mathscr{L}$, $T(dX|Y)$ is a probability measure on $\mathscr{K}' = \mathscr{P}(\mathscr{L}')$. (ii) For any finite $\Lambda \subset \mathscr{L}'$ and $A \subset \Lambda$,

$$T_\Lambda(A|Y) = T(I_\Lambda'(A)|Y) \tag{3.15}$$

is a continuous function of Y in $\mathscr{K}$. Here $I_\Lambda'(A)$ is the cylinder subset of $\mathscr{K}'$ defined in analogy with (3.2).

Theorem 3.5. Let $T(dX|Y)$ be a continuous conditional probability.
(i) If g is a continuous function on $\mathscr{K}'$, then

$$\int g(X)T(dX|Y)$$

is a continuous function of Y.

(ii) If μ is a measure on $\mathscr{K}$, the numbers

$$\mu_\Lambda'(A) = \int T_\Lambda(A|Y)\,\mu(dY) \tag{3.16}$$

satisfy the conditions (3.8) and (3.9) of Theorem 3.3 and hence correspond to a measure μ', which we write as

$$\mu'(dX) = \int T(dX|Y)\,\mu(dY) \tag{3.17}$$

on $\mathscr{K}'$. Furthermore, if μ is a probability measure, then so is μ'.

A sequence of continuous conditional probabilities T_j will be said to converge uniformly to a limit if for every finite Λ and A in Λ the sequence $T_{j\Lambda}(A|Y)$ converges to some function $T_\Lambda(A|Y)$ uniformly in Y (the convergence need not be uniform in Λ or A).

Theorem 3.6. If a sequence of continuous conditional probabilities T_j converges uniformly to a limit, the limiting $T_\Lambda(A|Y)$ corresponds to a (unique) continuous conditional probability $T(dX|Y)$. Furthermore, if μ_j is a sequence of measures on $\mathscr{K}$ converging to a limit μ, the sequence

$$\mu_j'(dX) = \int T_j(dX|Y)\,\mu_j(dY) \tag{3.18}$$

converges to the measure μ' given by (3.17).

The transformations introduced in Section 2 are special cases of the following product form. For each $k \in \mathscr{L}'$ let $t_k(Y)$ be a continuous, real-valued function on $\mathscr{K}$ with the property

$$0 \leqslant t_k \leqslant 1 \tag{3.19}$$

Define

$$T_\Lambda(A|Y) = \left[\prod_{k\in A} t_k(Y)\right] \prod_{k\in\Lambda\backslash A} [1 - t_k(Y)] \tag{3.20}$$

where $\Lambda\backslash A$ denotes the complement of A in Λ.

Theorem 3.7. The quantities $T_\Lambda(A|Y)$ defined in (3.20) uniquely determine a continuous conditional probability $T(dX|Y)$ through (3.15), provided the t_k satisfy (3.19) and are continuous.

4. INTERACTIONS AND EQUILIBRIUM EQUATIONS

4.1. Interactions and *W* Functions

A set of interactions Φ is defined to be a real-valued function on the finite subsets of $\mathscr{L}$. We shall usually assume that

$$\Phi(\varnothing) = 0 \tag{4.1}$$

although a finite value for this quantity merely shifts all energies by the same constant amount and hence does not affect the Gibbs probability distribution. The norm of Φ is defined by

$$\|\Phi\| = \sup_{i\in\mathscr{L}} \sum_{A:i\in A} |\Phi(A)| \tag{4.2}$$

which differs from the definition of Lanford and Ruelle[10] in that we do not assume that Φ is invariant under translations, and thus the sum may depend on the lattice site i. The $\Phi(A)$ are interpreted as dimensionless energies in lattice-gas language. For instance, if $A = \{i\}$, $\Phi(A)$ is $-\mu_i/kT$, where μ_i is the chemical potential at site i.

The dimensionless energy [the usual energy times $-(kT)^{-1}$] of a finite configuration B is equal to

$$U(B) = \sum_{A\subset B} \Phi(A) \tag{4.3}$$

In addition we define

$$W(A|B) = U(A \cup B) - U(B) = W(A\backslash B|B) \tag{4.4}$$

when A and B are finite. Even though $U(B)$ is undefined, in general, when B is an infinite set, it is often possible to define W in cases where its second argument is infinite. This includes the case in which $\|\Phi\|$ is finite.

Theorem 4.1. Suppose that

$$\|\Phi\| < \infty \tag{4.5}$$

Then for a fixed A, $W(A|B)$ as a function of its second argument satisfies the condition given in Theorem 3.2, and hence possesses a continuous extension to $\mathscr{X}$, which we shall denote by $W(A|X)$. This function satisfies the consistency condition

$$W(A \cup B|X) = W(A|B \cup X) + W(B|X) \tag{4.6}$$

where A and B are arbitrary finite subsets of $\mathscr{L}$, and X is any subset of $\mathscr{L}$.

The condition (4.5) turns out for some purposes to be unduly restrictive. It is also not invariant under particle–hole transformations (Section 4.3). Hence it is convenient to adopt the conclusions of Theorem 4.1 as a definition which is sufficient for most of the results of this section. We shall say that a function $W(A|X)$, whose first argument is always a finite set, is *consistent* if it satisfies (4.6) and *continuous* if for every A it is a continuous function of X. A sequence of functions W_j will be said to converge uniformly to a limit provided the limit

$$W(A|X) = \lim_{j\to\infty} W_j(A|X) \tag{4.7}$$

exists for every A and X and is uniform in X for a fixed A (but not necessarily uniform in A). The main properties of W functions which we shall need later are summarized in the following two theorems.

Theorem 4.2. If $W(A|X)$ is consistent, then

$$W(\varnothing|X) = 0 \tag{4.8}$$

for all X, If each W_j in a sequence is consistent and if the limit (4.7) exists, then the limit is consistent. If each W_j in a sequence is consistent and continuous and the sequence converges uniformly, then the limit is consistent and continuous. A consistent and continuous W can always be written in the form

$$W(A|X) = \sum_{\substack{B\subset A\cup X\\ B\cap(A\backslash X)\neq\varnothing}} \Phi(B) \tag{4.9}$$

where

$$\Phi(B) = (-1)^{|B|} \sum_{A\subset B} (-1)^{|A|} W(A|\varnothing) \tag{4.10}$$

and the sum in (4.9) is defined by choosing a sequence of finite sets C_n which converge to X, evaluating the right-hand side with X replaced by C_n, and taking the limit $n \to \infty$. If Φ defined by (4.10) has a finite norm [see (4.2)], then W coincides with the function whose existence is guaranteed in Theorem 4.1, and the right side of (4.9) is absolutely convergent.

Note that (4.9) and (4.10) together imply that a consistent and continuous $W(A|X)$ is determined uniquely by its values as a function of its first argument when $X = \varnothing$.

Theorem 4.3. If $W(A|X)$ is consistent and also a continuous function of X for every A with $|A| = 1$, then it is continuous for any finite A. If $W_j(A|X)$ is a sequence of consistent and continuous W functions with a limit, (4.7), which is uniform in X for every A for which $|A| = 1$, then the limit is uniform for every finite A, and the limit is consistent and continuous.

4.2. Equilibrium Equations

Given a finite set $A \subset \mathscr{L}$ and some $\Lambda \subset \Lambda$ as well as a measure μ on $\mathscr{K}$, we define a measure $\mu_\Lambda(A, dX)$ on

$$\mathscr{K}_\Lambda = \mathscr{P}(\mathscr{L}\backslash\Lambda) \tag{4.11}$$

through the requirement that

$$\mu_\Lambda(A, I_M{}^\Lambda(B)) = \mu(I_{\Lambda\cup M}(A \cup B)) \tag{4.12}$$

where M is a finite set in $\mathscr{K}_\Lambda$, $B \subset M$, and

$$I_M{}^\Lambda(B) = \{Y \in \mathscr{K}_\Lambda: \quad Y \cap M = B\} \tag{4.13}$$

The probability measure μ is said to be a *Gibbs state* with respect to a consistent and continuous $W(A|X)$ provided that the equilibrium equations

$$e^{-W(A|X)}\mu_\Lambda(A, dX) = \mu_\Lambda(\varnothing, dX) \tag{4.14}$$

are satisfied for every finite $\Lambda \subset \mathscr{L}$ and every $A \subset \Lambda$. That is to say, if g is any continuous function on $\mathscr{X}_\Lambda$,

$$\int g(X)e^{-W(A|X)}\,\mu_\Lambda(A, dX) = \int g(X)\,\mu_\Lambda(\varnothing, dX) \tag{4.15}$$

In view of the fact that $W(\varnothing\,|X)$ vanishes, (4.15) is equivalent to the assertion that the left side is independent of A for $A \subset \Lambda$.

Theorem 4.4. Let W_j be a sequence of consistent and continuous W functions converging uniformly to a limit W, and let μ_j be a Gibbs state with respect to W_j. If the μ_j tend to a limit μ [in the sense of (3.13)], then μ is a Gibbs state with respect to W.

The utility of this theorem is sometimes enhanced through the following observation:

Theorem 4.5. Let μ_j be a bounded sequence of measures on $\mathscr{X}$, that is, suppose there is a number $M < \infty$ such that

$$\mu_j(\mathscr{X}) < M \tag{4.16}$$

for all j. Then there is always a subsequence converging to a limit.

Theorem 4.6. Let $\mu(dY)$ be a Gibbs state with respect to a consistent and continuous W. Then

$$\exp[W(A|\,Y)] = \lim_{\Lambda\to\mathscr{L}} \mu_\Lambda((A \cup Y) \cap \Lambda)/\mu_\Lambda(Y \cap \Lambda) \tag{4.17}$$

and, in particular,

$$\exp[W(A|\varnothing)] = \lim_{\Lambda\to\mathscr{L}} \mu_\Lambda(A)/\mu_\Lambda(\varnothing) \tag{4.18}$$

The equilibrium equations may be thought of as a generalization to an infinite system of the usual prescription, (2.4), of probabilities for a finite system; that is, they provide a relationship between the interactions (or "Hamiltonian") embodied in Φ or W and the probability distribution represented by $\mu(dY)$. In contrast to a finite system, there may be more than one measure or equilibrium state associated with a set of interactions; this situation can arise at a phase transition (see Lanford and Ruelle[10]). However, given a Gibbs state, the corresponding interactions are uniquely determined.

It is also possible to write down the equilibrium equations in a fairly natural way for a finite system regarded as part of an infinite system (Section 4.4). Then Theorem 4.4 may be used to discuss the thermodynamic limit.

4.3. Particle–Hole Transformations

A *particle–hole transformation* on the set $R \subset \mathscr{L}$ may be thought of as replacing σ_i by $-\sigma_i$ for each $i \in R$, or as the one-to-one mapping of $\mathscr{P}(\mathscr{L})$ onto itself given by the symmetric difference:

$$Y \to \bar{Y} = R \,\Delta\, Y = (R \cup Y)\backslash(R \cap Y) \tag{4.19}$$

This mapping induces a natural transformation of functions:

$$f \to \bar{f}(Y) = f(R \,\Delta\, Y) \tag{4.20}$$

and of measures:

$$\mu \to \bar{\mu}; \qquad \bar{\mu}_\Lambda(A) = \mu_\Lambda(A \,\Delta(R \cap \Lambda)) \tag{4.21}$$

The transformation of a W function is a bit more complicated. The desired result is obtained using (4.4), assuming that the energy U transforms according to (4.20). Thus, as long as both R and X are finite sets, we can write

$$\bar{W}(A|X) = \bar{U}(A \cup X) - \bar{U}(X) = U(R \,\Delta(A \cup X)) - U(R \,\Delta X) \tag{4.22}$$

and hence

$$\begin{aligned}\bar{W}(A|X) = {} & W((A\backslash R)\backslash X|(R \,\Delta X)\backslash(R \cap A)) \\ & - W((A \cap R)\backslash X|(R \,\Delta X)\backslash(R \cap A))\end{aligned} \tag{4.23}$$

However, (4.23) also makes sense when R and X are infinite sets, so long as A is finite, and therefore we adopt it as a definition.

Theorem 4.7. Let W and $\bar{W}$ be related by (4.23). If W is consistent, $\bar{W}$ is consistent; if W is continuous, $\bar{W}$ is continuous. Furthermore, if μ is a Gibbs state with respect to a consistent and continuous W, $\bar{\mu}$ is a Gibbs state with respect to $\bar{W}$, and vice versa.

4.4. Finite System as Part of an Infinite System

For a finite system there is, of course, no ambiguity in defining the Gibbs probability. In lattice-gas notation the analog of (2.4), for a finite subset Ω of $\mathscr{L}$, is

$$\nu_\Omega(A) = e^{U(A)} \Big/ \sum_{B \subset \Omega} e^{U(B)} \tag{4.24}$$

If Λ is a subset of Ω and $B \subset \Lambda$, we define

$$\nu_\Lambda(B) = \sum_{\substack{A \subset \Omega \\ A \cap \Lambda = B}} \nu_\Omega(A) \tag{4.25}$$

It is convenient to extend ν to a measure μ^Ω on $\mathscr{K}$ defined by

$$\mu_\Lambda{}^\Omega(A) = 2^{-|\Lambda\backslash\Omega|} \nu_{\Lambda\cap\Omega}(A \cap \Omega) \tag{4.26}$$

This amounts to saying that each site outside Ω is occupied or vacant with probability 1/2, independent of what happens at other sites. Also, define W_Ω by

$$W_\Omega(A|X) = U((A \cup X) \cap \Omega) - U(X \cap \Omega) \tag{4.27}$$

which is to say that all interactions involving sites outside Ω vanish.

Theorem 4.8. The $\mu_\Lambda{}^\Omega$ defined by (4.26) correspond, in the sense of Theorem 3.3, to a probability measure μ^Ω satisfying (3.12). This measure is a Gibbs state with respect to W_Ω and is unique in the sense that it is the only probability measure satisfying (4.14) with W equal to W_Ω.

5. RESULTS AT LOW ACTIVITY

5.1. General Strategy

We shall show that if p is finite, a Kadanoff transformation (2.8) applied to an object system in which the lattice-gas activity is sufficiently small (or, equivalently, the Ising model magnetic field is sufficiently large) yields interactions for the image system which are well-defined in the thermodynamic limit and possess various pleasant properties: they decrease rapidly with distance and depend analytically on various parameters describing the object system.

The proof begins with an object system defined on a finite set of sites $\Omega \subset \mathscr{L}$. The W functions for the corresponding image system, denoted by $W_\Omega'(A|X)$, can be expressed in terms of thermal averages of operators involving a finite number of sites in a *modified* object system, in which the original object Hamiltonian has been somewhat altered. The state μ'^Ω of the image system is related to that of the object system μ^Ω by an equation of the form (3.18).

The equations of Gallavotti and Miracle-Sole are then used to show that under suitable conditions (low activity) μ^Ω tends to a limit μ and $W_\Omega'(A|Y)$ tends uniformly in Y to a limit $W'(A|Y)$ as Ω tends to $\mathscr{L}$. Theorem 3.6 then shows that μ'^Ω tends to a limit μ' given by (3.17). Since μ'^Ω is a Gibbs state with respect to W_Ω' (Theorem 4.8), it follows from Theorem 4.4 that μ' is a Gibbs state with respect to W', and from Theorem 4.6 that this W' is unique. Thus the renormalization transformation is unique in the thermodynamic limit.

The interactions Φ for the image system can be obtained from $W'(A|\varnothing)$ by means of (4.10). We derive explicit expressions for the interactions in terms of Ursell functions for the modified object system, and use these to show that Φ has a finite norm, (4.2), and various cluster properties. The low-activity condition also ensures that the interactions $\overline{\Phi}$ corresponding to $\overline{W}'$

obtained from W' using any particle–hole transformation, (4.19) and (4.23), also possess these pleasant properties.

5.2. Expression for W_Ω'

The Kadanoff transformation (2.8) for finite p can always be written in the form

$$T(\tau, \sigma) = \exp \sum_{j\in\Omega'} \left[p\tau_j \sum_{i\in C(j)} \sigma_i + \sum_{D\subset C(j)} Q_D \sigma_D \right] \tag{5.1}$$

where

$$\sigma_D = \prod_{i\in D} \sigma_i \tag{5.2}$$

and $\sigma_\varnothing$ is one. The Q_D are real constants chosen so that

$$\exp \sum_{D\subset C(j)} Q_D\sigma_D = \left[2\cosh\left(p \sum_{i\in C(j)} \sigma_i \right) \right]^{-1} \tag{5.3}$$

These formulas, along with those in Section 2, can be translated into the lattice-gas language of Sections 3 and 4 by noting that σ_i is the continuous function on subsets of $\mathscr{L}$ defined by

$$\sigma_i(X) = \begin{cases} +1 & \text{if} \quad i \in X \\ -1 & \text{otherwise} \end{cases} \tag{5.4}$$

and an analogous definition holds for τ_j as a function on $\mathscr{P}(\mathscr{L}')$. Thus $T_{\Omega'}(A \mid Y)$ in (3.20) is equal to the right side of (5.1) if the argument of the τ's is set equal to A and that of the σ's equal to Y, and, in particular

$$t_j(Y) = \exp\left[p \sum_{i\in C(j)} \sigma_i(Y) + \sum_{D\subset C(j)} Q_D\sigma_D(Y) \right] \tag{5.5}$$

Upon combining (2.1) and (5.1), we see that the (dimensionless) energy $U'(A)$ associated with a configuration $A \subset \Omega'$ is given by

$$\exp U'(A) = \mathrm{Tr}_\sigma \left\{ \exp\left[H(\sigma) + p \sum_{i\in C(A)} \sigma_i - p \sum_{i\in C(\Omega'\setminus A)} \sigma_i + \sum_{j\in\Omega'} \sum_{D\subset C(j)} Q_D\sigma_D \right] \right\} \tag{5.6}$$

where

$$C(A) = \bigcup_{j\in A} C(j) \tag{5.7}$$

Let $W_\Omega'(A \mid X)$ be defined by means of (4.27), with U and Ω replaced by U' and Ω' on the right side of that equation. Then with the help of (5.6) we obtain the formula, if $A \cap X = \varnothing$,

$$\exp W_\Omega'(A|X) = \left\langle \exp\left(2p \sum_{i \in C(A)} \sigma_i\right) \right\rangle_{\Omega, X} \tag{5.8}$$

where, if $\mathcal{O}(\sigma)$ is any function of the σ_i for i in Ω,

$$\langle \mathcal{O} \rangle_{\Omega, X} = \mathrm{Tr}_\sigma[\mathcal{O}(\sigma) \exp H_X(\sigma)]/\mathrm{Tr}_\sigma[\exp H_X(\sigma)] \tag{5.9}$$

and

$$H_X(\sigma) = H(\sigma) + p \sum_{i \in C(X \cap \Omega')} \sigma_i - p \sum_{i \in C(\Omega' \setminus X)} \sigma_i + \sum_{j \in \Omega'} \sum_{D \subset C(j)} Q_D \sigma_D \tag{5.10}$$

is the Hamiltonian for the *modified object system.* The final term on the right side of (5.10), which we shall call the "counter term," is a constant in the case of models I and II and can (for our purposes) be set equal to zero. The Hamiltonians $H(\sigma)$ and $H_X(\sigma)$ in (5.6), (5.9), and (5.10) refer to a set of sites Ω which is just $C(\Omega')$ in the case in which all the object sites lie inside some cell, and which includes additional sites nearby those in $C(\Omega')$ in cases, such as model II, in which not every object site is in a cell. [We shall, in particular, suppose that Ω is disjoint from $C(j)$ for any $j \notin \Omega'$, and that Ω tends to $\mathscr{L}$ as Ω' tends to $\mathscr{L}'$.]

5.3. Gallavotti–Miracle-Sole Equations

The lattice-gas correlation functions for a finite system $\Omega \subset \mathscr{L}$ are defined by

$$\rho_\Omega(A) = \langle n_A \rangle_\Omega \tag{5.11}$$

where

$$\langle \mathcal{O} \rangle_\Omega = \sum_{B \subset \Omega} \mathcal{O}(B) e^{U(B)} \Big/ \sum_{B \subset \Omega} e^{U(B)} \tag{5.12}$$

The energies $U(B)$ are determined by the interactions Φ through (4.3). The occupation variables are defined by

$$n_i(B) = \begin{cases} 1 & \text{for} \quad i \in B \\ 0 & \text{for} \quad i \notin B \end{cases} \qquad n_A = \prod_{i \in A} n_i \tag{5.13}$$

Gallavotti and Miracle-Sole[13] have shown that the correlation functions (5.11) satisfy certain linear equations. In the thermodynamic limit these equations can be written as a single vector equation

$$\rho = \alpha + K_\Phi \rho \tag{5.14}$$

on the Banach space $\mathscr{E}$ of bounded functions of finite subsets of $\mathscr{L}$, with the uniform norm equal to the supremum of the absolute value of the function. The vector α is given by

$$\alpha(A) = \begin{cases} z_i/(1 + z_i) & \text{if } A = \{i\} \\ 0 & \text{if } |A| \neq 1 \end{cases} \tag{5.15}$$

where

$$z_i = \exp \Phi(i) \tag{5.16}$$

is the activity at site i, and $\Phi(i)$ stands for $\Phi(\{i\})$.

Theorem 5.1. If $\|\Phi\| < \infty$, K_Φ is a bounded operator on $\mathscr{E}$ with norm

$$\|K_\Phi\| \leqslant \frac{|z| \exp\|\Phi_1\|}{1 + |z| \exp\|\Phi_1\|} [2 \exp(\exp\|\Phi_1\| - 1) - 1] \tag{5.17}$$

where

$$|z| = \sup_{i \in \mathscr{L}} |z_i| \tag{5.18}$$

and Φ_1 is obtained from Φ by setting the chemical potentials $\Phi(i) = 0$ for all $i \in \mathscr{L}$. If $\|K_\Phi\| < 1$, and in particular if

$$\frac{|z| \exp\|\Phi_1\|}{1 + |z| \exp\|\Phi_1\|} [2 \exp(\exp\|\Phi_1\| - 1) - 1] < 1 \tag{5.19}$$

then Eq. (5.14) has a unique solution in $\mathscr{E}$ given by

$$\rho(A) = \lim_{\Omega \to \mathscr{L}} \rho_\Omega(A) \tag{5.20}$$

More precisely, given any $\epsilon > 0$ and any finite set $A \subset \mathscr{L}$, there is a finite set Λ, which will in general depend on the interaction Φ, such that

$$|\rho_\Omega(A) - \rho(A)| < \epsilon \tag{5.21}$$

whenever $\Omega \supset \Lambda$. In particular, if (5.19) is satisfied, it is possible to choose Λ so that it is valid for the $|z|$ in question and also for all smaller values of $|z|$, with Φ_1 held fixed. The solution ρ is a real analytic function of the interactions Φ in the region (5.19).

Proof. The theorem is proved in a similar manner to Theorem 1 of Gallavotti and Miracle-Sole[13] (see also Ruelle: Ref. 8, p. 32; and Ref. 14) with obvious changes to allow for a lack of translational invariance.

Corollary 5.2. In the region (5.19) there is a unique Gibbs state μ (the limit as $\Omega \to \mathscr{L}$ of μ^Ω) for the infinite system.

Proof. This follows from Theorem 5.1 because each solution of the equilibrium equations yields a bounded solution of the Gallavotti–Miracle-Sole equations.[15]

Theorem 5.3. Let Φ be a set of interactions for an object system for which it is known that μ^Ω converges to a unique Gibbs state μ [e.g., assume that (5.19) is satisfied], and let $\hat{\Phi}$ be the interactions for the modified object system (5.10) in the case in which $X = \varnothing$ and Ω' tends to $\mathscr{L}'$. Then in the region

$$\frac{|\hat{z}| \exp(\|\hat{\Phi}_1\| + 4p)}{1 + |\hat{z}| \exp(\|\hat{\Phi}_1\| + 4p)} [2 \exp(\exp\|\hat{\Phi}_1\| - 1) - 1] < 1 \tag{5.22}$$

where $|\hat{z}|$ and $\hat{\Phi}_1$ are the analogs for the modified object system of the $|z|$ and Φ_1 previously defined for the object system, the Kadanoff transformation generated by (2.8) is well-defined and smooth in the thermodynamic limit, in the following sense:

a. The quantity $W_\Omega'(A|X)$ defined by (5.8) possesses a limit

$$W'(A|X) = \lim_{\Omega' \to \mathscr{L}'} W_\Omega'(A|X) \tag{5.23}$$

which is a continuous function of X and a real analytic function of the interactions Φ of the object system.

b. As $\Omega' \to \mathscr{L}'$, the interactions $\Phi_\Omega'(A)$ for the image system, which are related to $W_\Omega'(A|\varnothing)$ through (4.10), converge to limits $\Phi'(A)$, related to W' through (4.10), which are analytic functions of the interactions Φ.

c. As $\Omega' \to \mathscr{L}'$, the state μ'^Ω of the image system converges to a unique state μ' related to μ for the object system through (3.17), with $T(dX|Y)$ given by (3.20) and (5.5). Furthermore, μ' is a Gibbs state with respect to $W'(A|X)$; i.e., it satisfies (4.14).

Proof. The right side of (5.8) for a fixed A can be written as a sum of terms involving correlations of the form $\langle\sigma_B\rangle_{\Omega,X}$ with $B \subset C(A)$ or, equivalently, since σ_i is $2n_i - 1$, in terms of

$$\rho_\Omega(B; X) = \langle n_B\rangle_{\Omega,X} \tag{5.24}$$

The existence of a limit

$$\rho(B; X) = \lim_{\Omega \to \mathscr{L}} \rho_\Omega(B; X) \tag{5.25}$$

is a consequence of Theorem 5.1 when Φ is replaced by $\hat{\Phi}$. Note that H_X, (5.10), is related to $H_\varnothing$ by adding a term $2p\sigma_i$ for each i in $X \cap \Omega'$. This alters the activities at such sites by a factor of e^{4p}, but does not change $\hat{\Phi}_1$. Hence the extra $4p$ terms in (5.22) ensure that the condition (5.19) of Theorem 5.1 is satisfied for any choice of X, and that the limit (5.25), corresponding to (5.20), is uniform in X for a given B. However, since only those B that are subsets of the finite set $C(A)$ are in view, the limit is also uniform in B, and we conclude that the right side of (5.8), and thus the left, tends to a limit uniformly in X. This limit is bounded away from zero in view of the obvious inequalities:

$$\exp[-2p|C(A)|] \leqslant \exp\left(2p \sum_{i\in C(A)} \sigma_i\right) \leqslant \exp[2p|C(A)|] \tag{5.26}$$

and hence we obtain the uniform convergence of (5.23).

Theorem 5.1 also implies that the $\rho(B; X)$ in (5.25) are real analytic functions of the interactions $\hat{\Phi}$. But as the latter differ from Φ through the addition of certain terms [see (5.10)] dependent on p, it follows that the $\rho(B; X)$, hence also the $W'(A|X)$, are real analytic functions of the interactions Φ. The properties of Φ' in part (b) of Theorem 5.3 follow from those of W'. Finally, part (c) is proved through the chain of reasoning already discussed in Section 5.1.

5.4. Decay of Image Interactions

Let $H(\sigma)$ be a Hamiltonian, for a finite spin system Ω, with magnetic fields h_i. Then the Ursell functions are defined by

$$u_\Omega(A) = \left(\prod_{i\in A} \frac{\partial}{\partial h_i}\right) \log Z_\Omega(\mathbf{h}) \tag{5.27}$$

where

$$Z_\Omega(\mathbf{h}) = \mathrm{Tr}_\sigma \exp H(\sigma) \tag{5.28}$$

and $\mathbf{h}$ denotes a vector with components h_i. Thus, for example,

$$\begin{aligned} u_\Omega(\{i\}) &= \langle\sigma_i\rangle_\Omega \\ u_\Omega(\{i,j\}) &= \langle\sigma_i\sigma_j\rangle_\Omega - \langle\sigma_i\rangle_\Omega\langle\sigma_j\rangle_\Omega \end{aligned} \tag{5.29}$$

By Theorem 5.1 these Ursell functions have a well-defined thermodynamic limit

$$u(A) = \lim_{\Omega\to\mathscr{L}} u_\Omega(A) \tag{5.30}$$

in the region (5.19).

For transformations on finite systems the image interactions can always be obtained by straightforward calculation. Setting $X = \varnothing$ in (5.8), for the Kadanoff transformation (2.8) we obtain

$$\exp W_\Omega'(A|\varnothing) = \exp \sum_{S\subset A} \Phi_\Omega'(S) = \left\langle \exp\left(2p \sum_{i\in C(A)} \sigma_i\right) \right\rangle_{\Omega,\varnothing} \tag{5.31}$$

For simplicity consider model I. In this case

$$\left\langle \exp\left(2p \sum_{j\in A} \sigma_j\right) \right\rangle_{\Omega,\varnothing} = \hat{Z}_\Omega\left(2p \sum_{j\in A} \mathbf{e}_j\right) \Big/ \hat{Z}_\Omega(\mathbf{0}) \tag{5.32}$$

where

$$\hat{Z}_\Omega(\mathbf{h}) = \mathrm{Tr}_\sigma \exp\left[H_\varnothing(\sigma) + \sum_{i\in\Omega} h_i\sigma_i\right] \tag{5.33}$$

and $\mathbf{e}_j$ is a vector whose components all vanish except the jth component, which is unity. From (5.31) and (5.32) we find

$$\sum_{S \subset A} \Phi_\Omega'(S) = \log \hat{Z}_\Omega\Big(2p \sum_{j\in A} \mathbf{e}_j\Big) - \log \hat{Z}_\Omega(\mathbf{0}) \tag{5.34}$$

To obtain explicit expressions for the image interactions, the following result is now useful.

Lemma 5.4. If A is a finite set and $\Gamma(\mathbf{x})$ is a smooth function, then for arbitrary vectors $\mathbf{x}_j$,

$$\Gamma\Big(\sum_{j\in A} \mathbf{x}_j\Big) - \Gamma(\mathbf{0}) = \sum_{\varnothing \neq S \subset A} \int_0^1 \cdots \int \Big(\prod_{j\in S} \frac{\partial}{\partial t_j}\Big) \Gamma\Big(\sum_{j\in S} t_j \mathbf{x}_j\Big) \prod_{j\in S} dt_j \tag{5.35}$$

This result is easily proved by induction on $|A|$. Taking $\Gamma = \log \hat{Z}_\Omega$ and rewriting the right-hand side of (5.34), it readily follows that, for any $A \subset \Omega'$,

$$\Phi_\Omega'(A) = \int_0^{2p} \cdots \int \hat{u}_\Omega\Big(A \,\Big|\, \sum_{j\in A} t_j \mathbf{e}_j\Big) \prod_{j\in A} dt_j \tag{5.36}$$

where

$$\hat{u}_\Omega(A|\mathbf{h}) = \Big(\prod_{j\in A} \frac{\partial}{\partial h_j}\Big) \log \hat{Z}_\Omega(\mathbf{h}) \tag{5.37}$$

are the Ursell functions for the modified object system described by the Hamiltonian

$$H_\varnothing(\sigma) + \sum_{i\in\Omega} h_i \sigma_i \tag{5.38}$$

A similar formula holds for the general Kadanoff transformation. If $A \subset \Omega'$, and $B \subset \Omega$ consists of $|A|$ sites, one from each cell $C(j)$ for $j \in A$, we write $B \in C^A$. Using this notation, we find

$$\begin{aligned} \Phi_\Omega'(A) &= \int_0^1 \cdots \int \Big(\prod_{j\in A} \frac{\partial}{\partial t_j}\Big) \log \hat{Z}_\Omega\Big(2p \sum_{j\in A} t_j \sum_{i\in C(j)} \mathbf{e}_i\Big) \prod_{j\in A} dt_j \\ &= \sum_{B\in C^A} \int_0^{2p} \cdots \int \hat{u}_\Omega\Big(B \,\Big|\, \sum_{j\in A} \sum_{i\in C(j)} t_j \mathbf{e}_i\Big) \prod_{j\in A} dt_j \end{aligned} \tag{5.39}$$

Hence, using Theorem 5.3, we obtain the following result in the thermodynamic limit.

Theorem 5.5. Let Φ be the interaction for an object system $H(\sigma)$. Then in the region (5.22) the image interactions, obtained by applying the Kadanoff transformation (2.8) to the infinite object system, are given by

$$\Phi'(A) = \sum_{B\in C^A} \int_0^{2p} \cdots \int \hat{u}\Big(B \,\Big|\, \sum_{j\in A} \sum_{i\in C(j)} t_j \mathbf{e}_j\Big) \prod_{j\in A} dt_j \tag{5.40}$$

We will now show that at sufficiently low activity (large magnetic field) the image interactions (5.40) in fact (i) have finite norm and (ii) fall off exponentially rapidly with the geometrical size of the cluster formed by the sites in A if the object interactions are suitably short-ranged.

To show that the image interaction Φ' has a finite norm at sufficiently low activity, we need to consider configurations with multiple occupations. A configuration X on $\mathscr{L}$ is defined as a multiplicity function assigning a nonnegative integer $X(i)$ to each site $i \in \mathscr{L}$. In the sequel we will use X, Y, and Z to denote configurations, while retaining A, B, etc. to denote finite sets. This notation should not cause any confusion. The set of occupied points in the configuration X on $\mathscr{L}$ will be denoted by

$$\tilde{X} = \{i \in \mathscr{L}: \quad X(i) \geqslant 1\} \tag{5.41}$$

In addition, we set

$$|X| = \sum_{i \in \tilde{X}} X(i); \qquad X! = \prod_{i \in \tilde{X}} X(i)!; \qquad h^X = \prod_{i \in \tilde{X}} h_i^{X(i)} \tag{5.42}$$

and define the sum $X + Y$ of two configurations by

$$(X + Y)(i) = X(i) + Y(i) \tag{5.43}$$

The definitions (5.27)–(5.30) of the Ursell functions are extended by setting

$$u_\Omega(X) = (\partial/\partial h)^X \log Z_\Omega(\mathbf{h}) \tag{5.44}$$

and

$$u(X) = \lim_{\Omega \to \mathscr{L}} u_\Omega(X) \tag{5.45}$$

With these definitions the following result holds[13] (see Appendix B).

Lemma 5.6. Let Φ be an interaction with $\|\Phi\| < \infty$ and let

$$C(\Phi) = 2|z| \exp\|\Phi_1\| \exp(\exp\|\Phi_1\| - 1) \tag{5.46}$$

Then the Ursell functions (5.45) satisfy the cluster property

$$\sum_{\substack{X: i \in \tilde{X} \\ |X| = r}} |u(X)| \leqslant \frac{|z|}{C(\Phi)} \left(\frac{2C(\Phi)}{1 - C(\Phi)}\right)^r \tag{5.47}$$

for $r > 1$, in the low-activity region $C(\Phi) < 1$.

Theorem 5.7. Let Φ be the interaction for an object system $H(\sigma)$ with $\|\Phi\| < \infty$. Then the image interaction Φ', obtained by applying the Kadanoff transformation (2.8) to the infinite object system, has finite norm

$$\|\Phi'\| = \sup_{j \in \mathscr{L}'} \sum_{A \ni j} |\Phi'(A)| < \infty \tag{5.48}$$

in the low-activity region defined by

$$2|\hat{z}| \exp(\|\hat{\Phi}_1\| + 4p) \exp(\exp\|\hat{\Phi}_1\| - 1) < 1 \tag{5.49}$$

Here $\hat{\Phi}$ is the interaction for the modified object system (5.10) with Hamiltonian $H_{\varnothing}$.

Proof. Given a finite set $A \subset \mathscr{L}'$, consider the function

$$F(\mathbf{q}) = \sum_{B \in C^A} \int_0^{q_j} \cdots \int \hat{u}\left(B \,\middle|\, \sum_{j \in A} \sum_{i \in C(j)} t_j \mathbf{e}_i\right) \prod_{j \in A} dt_j \tag{5.50}$$

where $\mathbf{q}$ is a vector with components q_j if $j \in A$ and zero otherwise. If $q_j = 2p$ for all $j \in A$, $F(\mathbf{q})$ is just the image interaction $\Phi'(A)$, by Theorem 5.5. By Theorem 5.1 and (5.49), $F(\mathbf{q})$ is real analytic when $|q_j| \leqslant 2p$ for all $j \in A$. Its Taylor expansion at $\mathbf{q} = \mathbf{0}$ is

$$F(\mathbf{q}) = \sum_{r=0}^{\infty} \sum_{\substack{X : \bar{X} \subset A \\ |X| = r}} \left(\frac{\partial}{\partial q}\right)^X F(\mathbf{q}) \bigg|_{\mathbf{q}=0} \frac{q^X}{X!} \tag{5.51}$$

In the model I case, we expand

$$F(\mathbf{q}) = \int_0^{q_j} \cdots \int \hat{u}\left(A \,\middle|\, \sum_{j \in A} t_j \mathbf{e}_j\right) \prod_{j \in A} dt_j \tag{5.52}$$

$$= \sum_{r=|A|}^{\infty} \sum_{\substack{X : \bar{X} = A \\ |X| = r}} \left(\frac{\partial}{\partial q}\right)^{X-A} \hat{u}\left(A \,\middle|\, \sum_{j \in A} q_j \mathbf{e}_j\right) \bigg|_{\mathbf{q}=0} \frac{q^X}{X!} \tag{5.53}$$

where $X - A$ denotes the configuration given by

$$(X - A)(j) = \begin{cases} X(j) - 1 & \text{for } j \in A \\ X(j) & \text{for } j \notin A \end{cases} \tag{5.54}$$

Setting $q_j = 2p$ for all $j \in A$, and assuming convergence, we find

$$\Phi'(A) = \sum_{r=|A|}^{\infty} (2p)^r \sum_{\substack{X : \bar{X} = A \\ |X| = r}} \frac{\hat{u}(X)}{X!} \tag{5.55}$$

We now estimate

$$\begin{aligned} \|\Phi'\| &= \sup_{j \in \mathscr{L}} \sum_{A \ni j} |\Phi'(A)| \\ &\leqslant \sup_{j \in \mathscr{L}} \sum_{A \ni j} \sum_{r=|A|}^{\infty} (2p)^r \sum_{\substack{X : \bar{X} = A \\ |X| = r}} |\hat{u}(X)| \\ &\leqslant \sum_{r=1}^{\infty} (2p)^r \sup_{j \in \mathscr{L}} \sum_{\substack{X : \bar{X} \ni j \\ |X| = r}} |\hat{u}(X)| \\ &\leqslant 2p + \frac{|\hat{z}|}{C(\hat{\Phi})} \sum_{r=2}^{\infty} \left(\frac{4pC(\hat{\Phi})}{1 - C(\hat{\Phi})}\right)^r \end{aligned} \tag{5.56}$$

where we have interchanged the order of summation, and used Lemma 5.6

and the simple bound $|\hat{u}(X)| \leqslant 1$ when $|X| = 1$. But now observe that by (5.49)

$$C(\hat{\Phi}) = 2|\hat{z}| \exp\|\hat{\Phi}_1\| \exp(\exp\|\hat{\Phi}_1\| - 1) < \exp(-4p) \leqslant \frac{1}{1 + 4p} \tag{5.57}$$

Hence,

$$4pC(\hat{\Phi})/[1 - C(\hat{\Phi})] < 1 \tag{5.58}$$

and $\|\Phi'\| < \infty$ because (5.56) is a convergent geometric series. Notice also that by the above estimates the right-hand side of (5.55) is an absolutely convergent series, justifying (5.55) a posteriori.

In the general case the Taylor expansion is

$$F(\mathbf{q}) = \sum_{r=|A|}^{\infty} \sum_{\substack{X:\tilde{X}=A \\ |X|=r}} \sum_{B\in C^A} \left(\frac{\partial}{\partial q}\right)^{X-A} \hat{u}\left(B\left|\sum_{j\in A}\sum_{i\in C(j)} q_j \mathbf{e}_i\right)\right|_{\mathbf{q}=0} \frac{q^X}{X!} \tag{5.59}$$

Setting $q_j = 2p$ for all $j \in A$, we obtain

$$\Phi'(A) = \sum_{r=|A|}^{\infty} (2p)^r \sum_{\substack{X:\tilde{X}=A \\ |X|=r}} \sum_{B\in C^A} \sum_{Y\in C^{X-A}} \frac{\hat{u}(B + Y)}{X!} \frac{(X - A)!}{Y!} \tag{5.60}$$

where we write $Y \in C^X$ if Y is a configuration on $\mathscr{L}$ and X a configuration on $\mathscr{L}'$ and for each $j \in \mathscr{L}'$

$$\sum_{i\in C(j)} Y(i) = X(j) \tag{5.61}$$

From (5.60) we find

$$\Phi'(A) = \sum_{r=|A|}^{\infty} (2p)^r \sum_{\substack{X:\tilde{X}=A \\ |X|=r}} \sum_{Z\in C^X} \frac{|\hat{u}(Z)|}{Z!} \tag{5.62}$$

Hence finally, if $|C(j)|$ does not depend on $j \in \mathscr{L}'$, we estimate

$$\begin{aligned}
\|\Phi'\| &= \sup_{j\in\mathscr{L}'} \sum_{A\ni j} |\Phi'(A)| \\
&\leqslant \sup_{j\in\mathscr{L}'} \sum_{A\ni j} \sum_{r=|A|}^{\infty} (2p)^r \sum_{\substack{X:\tilde{X}=A \\ |X|=r}} \sum_{Z\in C^X} |\hat{u}(Z)| \\
&\leqslant \sup_{j\in\mathscr{L}'} \sum_{r=1}^{\infty} (2p)^r \sum_{\substack{X:\tilde{X}\ni j \\ |X|=r}} \sum_{Z\in C^X} |\hat{u}(Z)| \\
&\leqslant \sum_{r=1}^{\infty} (2p)^r \sup_{j\in\mathscr{L}'} \sum_{i\in C(j)} \sum_{\substack{Z:\tilde{Z}\ni i \\ |Z|=r}} |\hat{u}(Z)| \\
&\leqslant |C(j)| \sum_{r=1}^{\infty} (2p)^r \sup_{i\in\mathscr{L}} \sum_{\substack{Z:\tilde{Z}\ni i \\ |Z|=r}} |\hat{u}(Z)|
\end{aligned} \tag{5.63}$$

We conclude that $\|\Phi'\| < \infty$ as in the model I case.

To discuss the decay of interactions in more detail, we need some knowledge of the geometry of the lattices. We will assume that the lattice sites are located at points in Euclidean space. A graph is then a collection of straight lines between pairs of points on the lattice. The length of a graph is the sum of the lengths of its lines with respect to Euclidean distance (for a discussion of other distances see Duneau *et al.*[16]). A tree is a connected graph with no closed cycles. The geometrical size $L(A)$ of a cluster of sites A is the minimum of the lengths of all those trees that connect all the points of A *and* possibly arbitrary other points.[16] This is a more sensitive measure of the size of a cluster than the diameter, diam A, defined as the maximum of the lengths of all the lines joining pairs of points of A.

Theorem 5.8. Let Φ be the interaction for an object system $H(\sigma)$ and suppose that Φ is short-ranged, or more generally, that for some $\chi > 0$

$$\|\Phi\|_\chi = \sup_{i\in\mathscr{L}} \sum_{A\ni i} e^{\chi L(A)}|\Phi(A)| < \infty \tag{5.64}$$

Further suppose that the image lattice $\mathscr{L}'$ consists of sites from $\mathscr{L}$ selected one from each cell. Then in the low-activity region, defined by

$$2|\hat{z}| \exp(\|\hat{\Phi}_1\|_\chi + 4p) \exp(\exp\|\hat{\Phi}_1\|_\chi - 1) < 1 \tag{5.65}$$

the image interactions (5.40), obtained by applying the Kadanoff transformation (2.8) to the infinite object system, fall off exponentially rapidly with the geometrical size of the cluster A. Explicitly,

$$|\Phi'(A)| \leqslant (2pKM)^{|A|} e^{-\chi L(A)} \tag{5.66}$$

where

$$K = |C(j)| e^{\operatorname{diam} C(j)} \tag{5.67}$$

is a positive constant, independent of j because all the cells are assumed to have the same size and shape, and M is a positive constant depending on the interaction $\hat{\Phi}_1$.

Proof. Equation (5.40) holds by Theorem 5.5, because the region (5.22) contains the region (5.65). Duneau *et al.*[16] have shown that the Ursell function cluster property

$$|u(B)| \leqslant M^{|B|} e^{-\chi L(B)} \tag{5.68}$$

holds in the region $C(\Phi) < 1$, with M a positive constant depending on Φ_1. Moreover, the $4p$ term in (5.65) ensures that (5.68) holds for the Ursell functions $\hat{u}(B|\mathbf{h})$ uniformly in $\mathbf{h}$ over the entire range of integration in (5.40). Thus

$$|\Phi'(A)| \leqslant (2p|C(j)|)^{|A|} \max_{B\in CA} [M^{|B|} e^{-\chi L(B)}] \tag{5.69}$$

The result (5.66) now follows from the inequality

$$L(A) \leqslant \min_{B \in C^A} L(B) + |A| \operatorname{diam} C(j) \tag{5.70}$$

To prove this inequality, suppose the minimum on the right-hand side of (5.70) is attained for $B = B^*$. Now consider the tree of minimal length connecting the points of B^* and possibly other points and join each point in B^* with the point of A that lies in the same cell. This yields a new tree τ which now connects the points of A and whose length $L(\tau)$ satisfies

$$L(A) \leqslant L(\tau) \leqslant L(B^*) + |A| \operatorname{diam} C(j) \tag{5.71}$$

6. PECULIARITIES IN RENORMALIZATION-GROUP TRANSFORMATIONS

We shall now present evidence that certain position-space renormalization-group transformations applied to certain object Hamiltonians lack one or more of the properties (i)–(iii) listed in Section 1. The evidence for the existence of certain peculiarities in these transformations is quite compelling (though it falls slightly short of a rigorous proof). However, the precise nature of the peculiarities is somewhat obscure; various suggestions are considered in Section 6.3.

6.1. Model I

The most precise and unequivocal results are obtained in the case of model I with the object Hamiltonain given by (2.11) for sites on a square or simple cubic lattice. We are interested in the ferromagnetic case for which K is positive. It is known that the $\mu_\Lambda(A)$ [see (3.10)] for the equilibrium state are real analytic functions of h and K, provided $h > 0$ or $h < 0$, and for all h when K is sufficiently small ($K > 0$).[17] A phase transition occurs at $h = 0$ for K sufficiently large, and at the phase transition $\langle \sigma_A \rangle$ is a discontinuous function of h if $|A|$ is odd.[18] Here $\langle \cdots \rangle$ denotes an average with respect to the equilibrium state.

In model I each cell contains a single site. Upon identifying cell and site labels and formally taking the thermodynamic limit $\Omega \to \mathscr{L}$, (5.8) yields, for A a single site,

$$e^{W'(j|\varnothing)} = \langle e^{2p\sigma_j} \rangle_\varnothing = \cosh 2p + \langle \sigma_j \rangle_\varnothing \sinh 2p \tag{6.1}$$

where $\langle \cdots \rangle_\varnothing$ denotes an average with respect to the state corresponding to the Hamiltonian (5.10), with $X = \varnothing$, in the thermodynamic limit. In the case of model I, the counter term (involving Q's) in (5.10) is a constant, and thus $H_\varnothing$ is given by (2.11) with h replaced by $h - p$. Hence for K suffi-

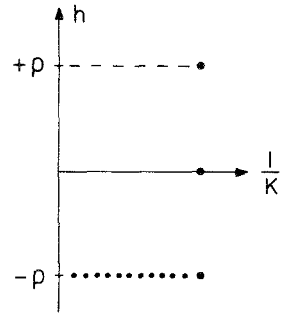

Fig. 1. Diagram showing where discontinuities arise in Eqs. (6.1) (dashed curve) and (6.2) (dotted curve.) The large dots are at $K = K_c$.

ciently large, $\langle\sigma_j\rangle_\varnothing$ will be *discontinuous* as a function of the parameter h at the value $h = p$ (the dashed line in Fig. 1), and thus $W'(j|\varnothing)$ will be discontinuous for this same h.

Note that this discontinuity takes place at a value of h for which the $\mu_\Lambda(A)$ for the object system and hence the $\mu_\Lambda'(A)$ for the image system are *analytic* functions of h and K. Hence if (6.1) is to be taken seriously one has a situation in which the W' (and hence also the Φ') associated with a state μ' exhibits singularities in a situation where the equilibrium state seems to show no anomalies. That is the reverse of what happens at an ordinary phase transition, in which the $\mu_\Lambda(A)$ exhibit singularities when the interactions, or W, vary smoothly, and suggests the name "anti phase transition" for the phenomenon under discussion. However, there are various possible explanations for the apparent unsmooth behavior of W' (see Section 6.3), and hence it seems best at present to retain the general term "peculiarity."

In addition, near the point $h = p$, $K = K_c$, where K_c is the critical value of K for the object system (the upper limit of the K values for which $\langle\sigma_j\rangle$ is continuous at $h = 0$), one expects $\langle\sigma_j\rangle_\varnothing$ and the $\langle\sigma_A\rangle_\varnothing$ for $|A| \geqslant 2$ to exhibit various "critical singularities" as functions of h and K (see, e.g., Refs. 7 and 19). These singularities will, of course, be reflected in the $W'(A|\varnothing)$. Furthermore, at $h = p$ and $K = K_c$ the Ursell functions decay slowly with distance and this will also be reflected in W' and the interactions for the image system. Hence (6.1) and its analogs for $|A| > 1$ certainly suggest violations of properties (ii) and (iii) of Section 1 in the case of model I.

But is (6.1) to be taken seriously? It represents a formal thermodynamic limit of (5.8) in a case in which we are not able to show that $W_\Omega'(A|X)$ converges uniformly (in X) to a limit, and hence cannot make use of Theorem 4.4. Indeed, it is precisely where (6.1) yields a discontinuous function of h that one must be most supicious about this convergence. Hence it may simply be that the transformation fails to exist in any well-defined sense.

It should be noted that the results of Section 5 imply that (6.1) *is* valid when h is sufficiently large. Hence, since the right-hand side is an analytic function for all $h > p$ and $h < p$, and for all h when K is sufficiently small, it is certain that the *analytic continuation* of the correct $W'(j|\varnothing)$ in the high-field region exhibits a certain amount of singular behavior in the region $h > 0$. What is not clear is the extent of the region in which this analytic continuation is equal to the corresponding term in a continuous $W'(A|X)$ associated with $\mu'(dY)$ through the equilibrium equations.

Additional peculiarities can be obtained by performing a particle–hole transformation for the image system with $R = \mathscr{L}$ (see Section 4.3). The analog of (6.1) then becomes

$$\exp[-W'(j|\mathscr{L}\backslash\{j\})] = \cosh 2p - \langle\sigma_j\rangle_{\mathscr{L}} \sinh 2p \tag{6.2}$$

where $H_{\mathscr{L}}$, which determines the state corresponding to the average $\langle\cdots\rangle_{\mathscr{L}}$, is given by (2.11) with h replaced by $h + p$. The expression (6.2) is analytic as a function of K and p for all $h > -p$ and $h < -p$; in particular along the dashed line in Fig. 1 where $W'(j|\varnothing)$ has discontinuities. However, (6.2) has discontinuities along the dotted line $h = -p$ in Fig. 1, at a location where (6.1) is analytic!

Next let X in (5.8) and (5.10) be the set $\mathscr{N}$ formed by one of the two interpenetrating sublattices with the property that all the nearest neighbors of the sites on one sublattice fall on the other. (It is only at this point in the argument that we need a square or simple cubic, in contrast to a triangular or face-centered cubic lattice), and let j be a site on the other sublattice. The analog of (6.1) is then

$$e^{W'(j|\mathscr{N})} = \cosh 2p + \langle\sigma_j\rangle_{\mathscr{N}} \sinh 2p \tag{6.3}$$

and the behavior of $\langle\sigma_j\rangle_{\mathscr{N}}$ must be determined by considering (5.10) with $X = \mathscr{N}$. This is the Hamiltonian for a system in a staggered magnetic field $+p$ on $\mathscr{N}$ and $-p$ on $\mathscr{L}\backslash\mathscr{N}$ in addition to the field h contained in $H(\sigma)$. By replacing σ_j by $-\sigma_j$ on $\mathscr{L}\backslash\mathscr{N}$, $H_{\mathscr{N}}$ is transformed into a system with antiferromagnetic interactions, K replaced by $-K$ in (2.11), uniform field $+p$, and a staggered field h with opposite signs on $\mathscr{N}$ and $\mathscr{L}\backslash\mathscr{N}$. For $h = 0$, such a Hamiltonian gives rise to a phase transition[20] of the antiferromagnetic type for values of K exceeding a function $K(p)$, with

$$K(p) \geqslant K_c \tag{6.4}$$

and equality only at $p = 0$. Translating these results back into the system described by $H_{\mathscr{N}}$ before the transformation, one concludes that the right side of (6.3) is, almost surely, a discontinuous function of h at $h = 0$ provided p is not too large and $K > K(p)$.

For a general X it is hard to say, especially since H_X lacks translational invariance, what "peculiarities" may arise in $W'(A|X)$ if one formally

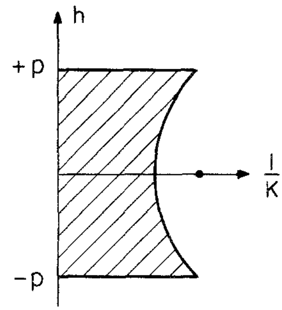

Fig. 2. Peculiarities may be anticipated in the transformation for model I within a region such as that shown by the cross-hatching.

takes the thermodynamic limit of (5.8) for model I. However, in view of the three special cases considered above, it is not unreasonable to suppose that there will be peculiarities of some sort in $W'(A|X)$, for some values of A and X, if the object Hamiltonian has parameters h and $1/K$ lying in a region similar to the shaded area in Fig. 2. This region is indented along the $1/K$ axis because of the observation that when $X = \mathcal{N}$ the peculiarities which occur at $h = 0$ arise when K exceeds $K(p)$, which is greater than K_c. It is, of course, possible that some other choice of X would give rise to peculiarities closer to the critical point at $h = 0$ and $K = K_c$, and in this sense Fig. 2, which is of course speculative, is also conservative: the region where peculiarities occur could be larger.

6.2. Other Transformations

6.2.1. General Strategy. The examples considered in the case of model I suggest the following strategy for producing peculiarities in other analogous renormalization-group transformations. One considers the modified object Hamiltonians produced by assigning definite values to each of the τ variables. If there is a set of parameters in the original object Hamiltonian which give rise to a phase transition in the state determined by the corresponding modified Hamiltonian, corresponding singularities can be expected, via the thermodynamic limit of (5.8), in the W' function for the image system—assuming, of course, that this procedure actually produces a W' which is, in some sense, to be associated with the state μ'.

6.2.2. Kadanoff Transformations with Small p. When p is small, it is plausible that the Kadanoff transformations give rise to a very similar situation as in the case of model I. The main difference is that the counter terms in (5.10) modify the simple Ising nearest-neighbor interactions in (2.11). However, since the Q_D are of order p^2 for small p and vanish unless $|D|$ is even, the basic "up–down', (σ_i replaced by $-\sigma_i$) symmetry of (2.11) for $h = 0$ is preserved, and hence the modified object Hamiltonians should give rise to phase transitions at the same values of h, though perhaps larger values of K, already considered in Section 6.1.

For larger values of p the counter terms can no longer be treated as a perturbation and the argument given above breaks down. It should be noted that whether p is to be considered "large" or "small" depends on the magnitude of K. Thus if p has some *fixed* finite value, no matter how large, one can presumably find a K value which is large enough that peculiarities will appear for $-p \leqslant h \leqslant p$. However, if p is allowed to increase as K increases—which is what is done in practice with Kadanoff transformations—the above argument *cannot* be used to argue for the existence of peculiarities.

Nevertheless, since peculiarities can also appear in the infinite-p limit (Section 6.2.4), it is by no means obvious that making p depend on K will prevent the Kadanoff transformation from running into difficulty.

6.2.3. Model II and the Decimation Transformation. The transformation T for model II is given in (2.10). The modified object Hamiltonian, analogous to (5.10), is $H(\sigma)$ plus a term

$$p \sum_j \pm \sigma_{(j)} \tag{6.5}$$

with the sign ($\pm$) in each case being that of the corresponding τ_j.

Consider the case in which all τ's are -1, K is large, and p is small. Then (6.5) amounts to a magnetic field $-p$ applied at those sites that are coupled to the τ's. It is then plausible that a phase transition will occur in the modified object system when h passes through a value *approximately* equal to $+p/c$, where c is the number of sites in a single cell, as in this case the total "average" field in a cell is zero. Thus for small p we expect model II to show qualitatively similar peculiarities to those discussed above for model I.

In the limit $p \to \infty$, model I yields the identity transformation, well-behaved but rather uninteresting. Model II in this limit is the "decimation transformation." Each $\sigma_{(j)}$ has precisely the same value as the corresponding τ_j and is no longer a dynamical variable in the modified object Hamiltonian. It produces, however, via the first term in (2.11), a magnetic field of magnitude $\pm K$ on its nearest neighbors.

Consider the case in which the cells are relatively large squares or cubes (in two or three dimensions, respectively), and all the τ's have the value -1. The modified object Hamiltonian refers to a lattice from which a small fraction of the sites have been deleted, and the interactions include an additional magnetic field $-K$ on some of the remaining sites, which are again a small fraction of the total. Since this extra field is applied at only a few sites, it is plausible that its effect can be compensated by a uniform h so that the modified object system undergoes a phase transition when h is positive but of magnitude much less than K, a transition in which $\langle\sigma_j\rangle$ is discontinuous if j is *not* one of the special sites (j). Such a phase transition leads to a discontinuity in the formal expression for W' (see Appendix C), even though the analog of (5.8) cannot be used directly.

While the existence of a similar phase transition in the case of a small cell is by no means obvious, it should be noted that iterating the model II transformation a number of times is equivalent to carrying it out once on a larger cell with a modified p value. If the initial p is infinite, the modified p is also infinite. Hence if peculiarities are absent the first time the transformation is applied, they may well appear when the transformation has been iterated a few times.

This last point can be illustrated in terms of the square lattice with Hamiltonian (2.11). If the $\sigma_{(j)}$ form the sublattice (the squares of one color on a checkerboard), the decimation transformation can be carried out once exactly, and the resulting transformed Hamiltonian has interactions of short range.(5,6) However, carrying out this transformation twice is equivalent to a single transformation in which the $\sigma_{(j)}$ form a square lattice with twice the lattice constant of the original lattice. In this case an antiferromagnetic choice for the τ_j—opposite signs on nearest-neighbor sites on $\mathscr{L}'$—leads to a modified object system Hamiltonian which is equivalent to that of a square lattice,[5] (2.11), with altered coupling constants, and hence can be shown, quite rigorously, to possess a phase transition at $h = 0$ and K sufficiently large.

6.2.4. The Niemeijer and van Leeuwen Transformation on a Triangular Lattice. If the parameter p in the Kadanoff transformation, (2.8) or (5.1), is allowed to go to infinity, T remains well defined and the result

[5] The modified object system Hamiltonian is that of a "decorated" Ising model, which can be transformed into an ordinary square model by standard techniques. See, for example, Syozi.(21)

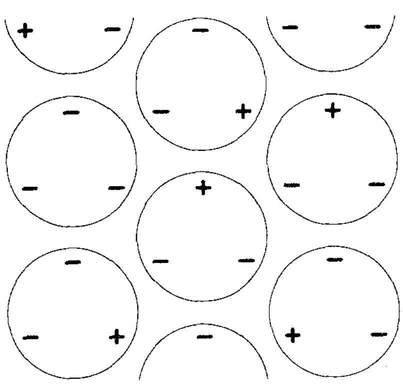

Fig. 3. Part of a possible configuration on a triangular lattice when the τ's associated with the cells indicated by circles are all -1. Values of the σ's at the different sites are indicated by plus and minus signs.

is a transformation of the Niemeijer and van Leeuwen or "majority rule" type.[4] The situation is perhaps easiest to visualize in the case of a triangular lattice. We shall assume that the object system Hamiltonian is of the form

$$H(\sigma) = K \sum_{\langle ij \rangle} \sigma_i \sigma_j + K' \sum_{\{lm\}} \sigma_l \sigma_m + h \sum_j \sigma_j \tag{6.6}$$

where $\langle ij \rangle$ stands for a nearest-neighbor and $\{lm\}$ a *next-nearest-neighbor* pair of sites. A cell consists of three nearest-neighbor sites forming an equilateral triangle (see Fig. 3). The transformation $T(\tau, \sigma)$ vanishes unless in each cell the τ variable for this cell has the same sign as the sum of the σ variables in the cell. When this constraint is satisfied in every cell, T has the value 1.

Consider the modified object Hamiltonian obtained by requiring that all the τ's be -1. This amounts to saying that the only configurations of the σ's that need be considered are those for which $\sigma_i = -1$ for at least two of the sites in each cell. Figure 3 illustrates a possible configuration. The energy of an allowed configuration is determined by (6.6) with no additional terms.

Let us assume that h is very large and positive. Then with high probability there will be one σ_i equal to $+1$ in each cell. Thus each cell has effectively three different states. If $K' = 0$, the maximum $H(\sigma)$ under these circumstances corresponds to a "disordered ground state" in which a very large number of configurations have equal energy. If, however, K' takes on a small positive or negative value, the maximum $H(\sigma)$ occurs for a much smaller number of "ordered" ground states, each of which possesses a lower symmetry than the modified object Hamiltonian (i.e., is invariant under a subgroup of the translations and rotations which map the cells into each other). One's general experience with systems in which the ground state possesses a broken symmetry suggests that if K and K' decrease from very large values, with K'/K fixed, or if K and K' are fixed and h decreases, the equilibrium state will eventually change from one of broken symmetry to one with the same symmetry as the Hamiltonian, with this change taking place at a well-defined phase transition. If such is the case we would, on the basis of the argument in Appendix C, anticipate a peculiarity in the corresponding W'.

On the other hand, if $h = 0$, the modified object system with all τ's equal to -1 has a ground state—assuming $K > 0$ and $|K'| \ll K$—in which $\sigma_i = -1$ for all i. In this case there is no broken symmetry and no reason to expect a phase transition for any value of K. Unfortunately, this does not mean that W' lacks peculiarities; they may well appear for some alternative choice of τ's. Our investigation of model I suggests that some arrangement of τ's in which half are equal to $+1$ and half are equal to -1 is probably a good place to look for peculiarities at $h = 0$.

6.3. Comments on the Peculiarities

Unfortunately, for reasons discussed in Section 6.1, it is difficult to demonstrate rigorously from the existence of singularities in the transformations defined formally in the thermodynamic limit that the "real" W' possesses such singularities. It is, in our opinion, most unlikely that there exist transformations satisfying properties (i)–(iii) of Section 1 in regions where the peculiarities make their (formal) appearance. However, the list of possible ways in which things can be going wrong is, unfortunately, fairly large. Some possibilities are:

1. There exists no W' which can be associated with μ' through equilibrium equations.
2. There exists a $W'(A|X)$ which is associated with μ' through the equilibrium equations, but it is a discontinuous function of its second argument.
3. There exists a W' associated with μ' through the equilibrium equations, but it is not correctly given by taking the thermodynamic limit of (5.8).
4. The W' associated with μ' through the equilibrium equations is correctly given by the limit of (5.8) and is continuous, but is not a smooth function of the parameters in the object Hamiltonian.

If the first possibility is correct, there is no way of defining a transformation using the method adopted in this paper. Possibility 3 seems unlikely, though we cannot at present disprove it. But if true, it raises the troublesome issue of the physical significance of a W' that does not correspond to a thermodynamic limit. Although the thermodynamic limit, in the sense of allowing a finite system to become infinite, is not essential for discussing infinite systems, it would seem to be a useful tool for singling out those features of infinite systems that are relevant to experiments carried out in the laboratory. In any case, our results for model I give us no reason to suppose that such a W' would be better behaved than the expressions obtained from the thermodynamic limit of (5.8).

The second possibility seems to us a plausible explanation in those cases where the modified object system undergoes a first-order phase transition as a function of a parameter in the object system: models I and II when p is small, and the decimation transformation. True enough, discontinuity of $W'(A|X)$ as a function of h is no guarantee of discontinuity as a function of X, but it is not implausible that the two should be related. Whereas our discussion of equilibrium equations is limited to the case of a continuous W, there seems to be no reason why these equations cannot be generalized to allow for a discontinuous W. Unfortunately, there is a price to be paid for such a generalization. Theorem 4.6 is no longer valid and one cannot in general expect that a unique (discontinuous) W will be associated with a (generalized) Gibbs state. Even if this problem can be surmounted, Theorem

4.2 no longer applies and one cannot in general associate a set of interactions with a discontinuous W. Thus if it results in a discontinuous W', it is hard to see how one can regard a renormalization-group transformation as mapping a Hamiltonian onto a Hamiltonian. Of course there may be ways to circumvent these difficulties, but they obviously require further research.

Possibility 4 seems plausible in those cases in which the "peculiar" phase transitions are continuous ("second-order") rather than discontinuous—as may well be the situation in the case of the Neimeijer and van Leeuwen transformation on a triangular lattice. Only further investigation will permit one to decide between 2 and 4, assuming that one of them is correct.

As a final comment we note that each of the "peculiarities" discussed above has the property that it arises from a phase transition, in a modified object system, of a sort which seemingly has *nothing whatever* to do with the physically important features of the equilibrium state for the image system. Thus in model I the modified object system undergoes a ferromagnetic transition when the image system is far from its own ferromagnetic transition. On the other hand, when $h = 0$, we generated a "peculiarity" in the modified object system by employing an *antiferromagnetic* arrangement of τ's. The same sort of observation is valid for the transformations in Section 6.2.

This peculiar behavior of the "peculiarities" suggests to us that their origin may in some sense lie in the demand that H' in (2.7) generate ρ' *exactly*, including the correct probabilities for configurations that are quite unlikely and hence, for most physical applications, unimportant. And it raises the intriguing question as to whether the peculiarities we have uncovered could not be "cured" by the device of modifying ρ' in a manner which makes minor alterations in the probabilities of "likely" configurations and major alterations in the probabilities of "unlikely" configurations while still keeping the latter small.

7. CONCLUSIONS

Our calculations establish two important results for certain position-space renormalization-group calculations:

1. Kadanoff transformations with finite p have all of the desirable properties listed in Section 1 when the activity of the object system is sufficiently small (or the magnetic field sufficiently large, in magnetic language): The transformation is well-defined, and the image interactions are analytic functions of the (real) parameters which appear in the interactions for the

object system, and fall off rapidly with distance provided the object interactions are of sufficiently short range. We have not been able to show that there is a region of the parameter space which maps *into itself* under a Kadanoff transformation; the basic difficulty is that a small activity in the object system does not guarantee a small activity in the image systems.

2. There is compelling evidence that Kadanoff transformations, decimation transformations, and Niemeijer and van Leeuwen transformations exhibit "peculiar" behavior for certain classes of object-system Hamiltonians. These peculiarities seem to have nothing to do with phase transitions of the object or image states, in the sense that they can occur in cases where the $\mu_\Lambda(A)$ and $\mu_\Lambda'(A)$ for the object and image systems, respectively, are analytic functions of the parameters in the object-system Hamiltonian. Although their precise significance is obscure (see Section 4.1), they almost certainly indicate violations of one or more of the desirable properties listed in Section 1, and may in particular imply that a renormalization-group transformation, as a map of Hamiltonians onto Hamiltonians, is not defined (for certain object Hamiltonians) in the thermodynamic limit.

The overall significance of these results for the renormalization-group enterprise in statistical mechanics is not easy to assess. It may assist the reader if we set forth two opposing viewpoints, optimistic and pessimistic, both of which are compatible with the results of this paper.

The optimist will note that we have demonstrated the existence of a nontrivial class of interactions for which at least one type of position-space renormalization-group transformation is both well-defined and well-behaved. Thus it is certainly not possible for even the most skeptical theoretical physicist to dismiss the whole enterprise as mathematical nonsense. Furthermore, it is reasonable to suppose that the range of applicability of these transformations is much wider than that in which they can be proved to give good results. The instances in which peculiarities arise fall in limited regions of the parameter space for the object system Hamiltonians, and if Fig. 2 is typical, this region need not include the critical points where renormalization-group methods have been particularly useful. In addition, it is quite possible that the peculiarities arise from the very-many-body interactions which are always ignored in practical calculations. Thus the latter, at least, need not be afflicted with the misleading pathologies which can arise for exact transformations.

The pessimist will note that the only cases in which we are certain that renormalization-group techniques work well are in the regime of low activity, where such methods are not really needed. In other parts of the Hamiltonian space, including those in which phase transitions arise, there is reason to suspect that renormalization transformations are infested with peculiar pathologies having no sensible connection with the physics of the object

Hamiltonian, and hence likely to make their unwelcome appearance where the theoretician least expects them. These peculiarities strike at the very heart of renormalization-group phenomenology: the notion that there are smooth transformations "from Hamiltonians to Hamiltonians." Even if the first application of a transformation is on firm ground, the examples of Section 6 suggest that the second, third, or nth step of iterating the transformation may still lead into quicksand. The fact that these pathologies do not seem to have been reported for the approximate transformations used in practice is good reason to suppose that such approximations are misleading in subtle, and perhaps less subtle, respects. Indeed, the whole renormalization-group procedure may be no more than another (to be sure, very successful) phenomenological approach to phase transitions, rather than, as some enthusiastic practitioners would have us believe, the genuine bridge between microscopic models and macroscopic physics.

There is a third position, intermediate between the optimistic and pessimistic, which has something to commend it. We suggested in Section 6.3 that peculiarities might arise in exact transformations due to the demand that the image Hamiltonians reproduce the exact probability distribution of the image state. Relaxing this demand, and in particular only requiring that the probabilities be given (nearly) correctly for the more likely or typical configurations, might, perhaps, remove the peculiarities. Since the approximate transformations employed in practice do not (so far as we know) exhibit the peculiarities we have discussed and since they, as a matter of necessity, ignore interactions involving spins on a large number of sites, it may well be that they represent a practical realization of the program suggested in the preceding sentence.

If this is the case, one can understand why the very good results obtained in actual applications of position-space renormalization-group methods are not in contradiction with the results obtained in Section 6. These methods could, perhaps, be viewed as analogs of asymptotic series; further refinements in the approximations might, after some stage, fail to yield better results. But if this is so, it is also possible that some of the strong *general* conclusions which renormalization-group methods yield, such as the "strong scaling" connection between certain critical exponents and dimensionality, could be in error. Naturally, all of these suggestions are at present quite speculative.

In conclusion, we wish to comment on two additional points. First, while our calculations say nothing about the existence and properties of renormalization-group transformations which involve integrating out degrees of freedom in *momentum* space, they do indicate the importance of a precise mathematical investigation of these procedures to test their range of validity.

The second point is one which has often come up during informal discussions of our results and may well have occurred to the reader: Since all the really difficult problems in the chain of transformations indicated in (2.7) arise when trying to construct H' from ρ', why not simply restrict the whole renormalization-group enterprise to a discussion of transformations from states to states,[22] ρ to ρ' to ρ'', etc., and never discuss Hamiltonians?

This is certainly a valid question, and our response is as follows: While it *may* be possible to set up the whole renormalization-group machinery of flows, eigenoperators near fixed points, and the like in a suitable space of states, rather than interactions, the task is far from trivial. To take just one example, the critical exponents α, β, and ν (among others) are defined in terms of "temperature differences" from the critical point and whereas a temperature difference is a fairly well-defined notion in the space of interactions, we, at least, do not know how to define it or an analogous quantity in the space of states.

The naive notion that there must be a simple connection between formalisms worked out for spaces of interactions and spaces of states seems to be based on the comparative simplicity of the formula, (2.4), that relates the two for finite systems. This simplicity vanishes, however, in the thermodynamic limit, and its disappearance is, in a sense, precisely what permits one to discuss in mathematical terms a rich variety of phase transitions. Thus, while we would not at all wish to deny the potential value of an approach based solely on transformations from states to states, it is at present the name of an uncompleted research project rather than an immediate solution to the problems discussed in this paper.

APPENDIX A. REFERENCES AND PROOFS FOR THE RESULTS OF SECTIONS 3 AND 4

Various facts about the topology of $\mathscr{K}$, states of infinite lattice systems, equilibrium equations, etc., are succinctly set forth in the papers of Lanford and Ruelle.[10,11] The remarks in this appendix are purely for the benefit of physicists who, like ourselves, find this sort of mathematics unfamiliar, and may wish to know what references we used and how we thought about these problems.

For topological concepts, we employed Kelley's *General Topology*[23] (page numbers are given in parentheses): Since the topology of $\mathscr{K}$ satisfies the second, and hence also the first, axiom of countability (pp. 48, 50), it can be conveniently characterized by sequences (p. 72). Compactness can be checked directly, and is also a consequence of Tychonoff's theorem (p. 143) applied to $\mathscr{K} = \{0, 1\}^{\mathscr{L}}$.

For any infinite set X there is always a sequence of finite sets A_j converging to X, and hence a continuous function is uniquely determined by its values on finite sets. The other assertions in Theorem 3.1 are proved using compactness in a manner parallel to the corresponding proofs for functions on compact sets of real numbers. The conditions given in Theorem 3.2 mean that $f(A_j)$ form a Cauchy sequence for a sequence A_j converging to X, and can be used to show that the limit is unique (independent of the choice of approximating sequence) and continuous.

We used Halmos' *Measure Theory*[24] as the basis for Section 3.2 (page numbers are given in parentheses): Subsets of $\mathscr{X}$ made up of finite unions of mutually disjoint cylinder sets form a Boolean ring $\mathscr{R}$ (p. 19) on which a function μ is defined as follows: The value

$$\mu(I_\Lambda(A)) = \mu_\Lambda(A) \tag{A.1}$$

is assigned to the cylinder set $I_\Lambda(A)$, and μ is made additive for finite unions of disjoint cylinder sets. Condition (3.9) ensures the consistency of this prescription. The result is a finite measure (pp. 30, 31), since

$$\mu(\mathscr{X}) = \sum_{A\subset\Lambda} \mu(I_\Lambda(A)) \tag{A.2}$$

is [by (3.8)] finite. It possesses (p. 54) a unique extension to a measure μ on the σ-ring generated by $\mathscr{R}$. Let Ω_j be an increasing ($\Omega_j \subset \Omega_{j+1}$) sequence of finite subsets converging to $\mathscr{L}$. If f is a continuous function, the sequence of simple (p. 95) functions f_j defined by

$$f_j(X) = f(X\cap\Omega_j) \tag{A.3}$$

with integrals given by

$$\int f_j(X)\,\mu(dX) = \sum_{A\subset\Omega_j} f(A)\mu_{\Omega_j}(A) \tag{A.4}$$

converges to f (pointwise), and since the latter is bounded (Theorem 3.1), it is integrable (p. 110), and its integral is the limit of (A.4) as $j\to\infty$. This result justifies (3.11). Note that (3.11) is not (in general) valid for discontinuous integrable functions.

Theorem 3.4 may be proved as follows. Since $\mu(\mathscr{X})$ is finite, there is a finite number M such that $\mu_j(\mathscr{X})$ is less than M for all j. Hence, since f_j converges uniformly to f, given any $\epsilon > 0$ there will be a J such that $j \geqslant J$ implies that

$$\left|\int f_j\,\mu_j(dY) - \int f\,\mu_j(dY)\right| \leqslant \int |f_j - f|\,\mu_j(dY) < \epsilon M \tag{A.5}$$

Now f itself is uniformly continuous by Theorem 3.1, and therefore there is a Λ such that if $X\cap\Lambda$ is the same as $Y\cap\Lambda$, $f(X)$ and $f(Y)$ can differ by at most ϵ. This fact together with the consistency condition (3.9) can be used to show that for any larger $M \supset \Lambda$,

$$\left|\sum_{B\subset M} f(B)\mu_{jM}(B) - \sum_{A\subset\Lambda} f(A)\mu_{j\Lambda}(A)\right| < \epsilon M \tag{A.6}$$

Consequently, in view of (3.11), for any $j \geqslant J$ the integral on the right side of (3.14) can differ from that on the left by at most $2\epsilon M$.

To prove Theorems 3.5(i), one uses the fact that g is uniformly continuous to show that the sums approximating the integral, in analogy with (3.11), converge uniformly to a function which, by Theorem 3.1, must be continuous. Part (ii) of this theorem is straightforward, as is the first part of Theorem 3.6. The second part of Theorem 3.6 is proved by applying Theorem 3.4 to

$$\mu'_{j\Lambda}(A) = \int T_{j\Lambda}(A \mid Y)\, \mu_j(dY) \tag{A.7}$$

for each Λ and each $A \subset \Lambda$.

In proving Theorem 3.7, note that the continuity of $T_\Lambda(A \mid Y)$ in (3.20) is an obvious consequence of the continuity of the t_k. For Y fixed, the analog of (3.8) is obvious, while that of (3.9), because of the simple product form of (3.20), can be transformed into

$$T_M(B \mid Y) = \sum_{C \subset \Lambda \setminus M} T_M(B \mid Y) T_{\Lambda \setminus M}(C \mid Y) \tag{A.8}$$

Hence it suffices to show that if t_k are any real numbers, and Λ any finite set,

$$\sum_{C \subset \Lambda} \left(\prod_{k \in C} t_k \right) \prod_{k \in \Lambda \setminus C} (1 - t_k) = 1 \tag{A.9}$$

a result easily established by induction on $|\Lambda|$. [Note that the sums in (A.8) and (A.9) include $C = \varnothing$.]

Theorem 4.1 may be proved with the help of the following:

Lemma A.1. Given some finite set A and some $\epsilon > 0$, there is a finite set Λ such that

$$\sum_{\substack{C: C \cap A \neq \varnothing \\ C \not\subset \Lambda}} |\Phi(C)| < \epsilon \tag{A.10}$$

provided (4.5) is satisfied.

Proof. Given (4.2) and (4.5) we can evidently find, for each $i \in A$, a Λ^i such that

$$\sum_{\substack{C: i \in C \\ C \not\subset \Lambda_i}} |\Phi(C)| < \epsilon / |A| \tag{A.11}$$

One then sets Λ equal to the union of the Λ_i and observes that each summand in (A.10) also occurs in (A.11) for some $i \in A$.

Naturally, the Λ of the lemma may be chosen to include A. If this is done, and if (3.6) holds for X and Y replaced by B and B', respectively, a brief computation shows that

$$|W(A \mid B) - W(A \mid B')| < \epsilon \tag{A.12}$$

because when the difference of the W's is written out using (4.4) and (4.3), it is seen to consist of a sum of terms, each of which appears on the left side of (A.10). Thus the condition of Theorem 3.2 is satisfied. The condition (4.6) follows from (4.4) when X is finite, and by continuity when X is infinite.

In Theorem 4.2, (4.8) is a consequence of (4.4). The consistency of the limit (4.7) is straightforward; its continuity is a consequence of Theorem 3.1. One may use (4.10) to define Φ if W is consistent and continuous, and the Möbius inversion formula[25] yields

$$W(A|\varnothing) = \sum_{B\subset A;B\neq\varnothing} \Phi(B) \tag{A.13}$$

Next, the expression

$$W(A|C) = W(A\cup C|\varnothing) - W(C|\varnothing) \tag{A.14}$$

which is a special case of the consistency condition (4.6), may be used together with (A.13) to establish (4.9) when $X = C$ is finite. The result for infinite X follows by continuity, but since the sum in (4.9) is not in general absolutely convergent, it is necessary to specify in what sense it exists. If, however, (4.5) is satisfied, the absolute convergence of (4.9) follows from Lemma A.1.

Theorem 4.3 is a consequence of observing that repeated applications of the consistency condition (4.6) permit any $W(A|X)$, A of course finite, to be expressed as a finite sum of terms of the form $W(\{j\}|X_j)$, with $j\in\mathscr{L}$, and X_j an appropriate subset of $A\cup X$.

The proof of Theorem 4.4 is obtained by applying Theorem 3.4 to both sides of (4.15) with μ replaced by μ_j. It is easy to show that the convergence of μ_j to μ implies the convergence of $\mu_{j\Lambda}(A, dY)$ to $\mu_\Lambda(A, dY)$, and that

$$g(Y)\exp W_j(A|Y)\to g(Y)\exp W(A|Y) \tag{A.15}$$

uniformly in Y.

Theorem 4.5 is a consequence of an "elementary compactness argument"[11] which may be constructed as follows. As there are a countable number of pairs (Λ, A) in which Λ is a finite subset of $\mathscr{L}$ and $A\subset\Lambda$, we can number them by the positive integers. Since the $\mu_{j\Lambda}(A)$ for the first pair (Λ, A) lie between 0 and M for every j, one can select a subsequence which converges to a number in this same interval. For the second pair a similar selection process is carried out with, however, j limited to the subsequence obtained when considering the first pair. The selection process continues in a similar manner, and, evidently, the sequence consisting of the first element of the first subsequence, the second element of the second subsequence, the third of the third, etc., converges for every (Λ, A).

To obtain (4.18), set g equal to one in (4.15), so that

$$\begin{aligned}\mu_\Lambda(A) &= \int e^{W(A|Y)}\mu_\Lambda(\varnothing, dY) \\ &= e^{W(A|\varnothing)}\mu_\Lambda(\varnothing)\cdot\left[\int e^{W(A|Y)-W(A|\varnothing)}\mu_\Lambda(\varnothing, dY)\Big/\int \mu_\Lambda(\varnothing, dY)\right]\end{aligned} \tag{A.16}$$

The Y which appears as the second argument of W in (A.16) is a subset of $\mathscr{L}\backslash\Lambda$. Hence, since W is uniformly continuous in this argument, we can make $W(A|Y) - W(A|\varnothing)$ as small as desired by choosing Λ sufficiently large. Thus as Λ tends to $\mathscr{L}$, the quantity in square brackets in (A.16) approaches one, which establishes (4.18). The general case (4.17) is established in the case $A \cap Y = \varnothing$ (no loss of generality) by using a particle–hole transformation and Theorem 4.7 in the case $R = Y$.

In Theorem 4.7, the continuity of $\overline{W}$ is an immediate consequence of (4.23), while consistency can be checked by a straightforward, though laborious, calculation using appropriate forms of (4.6). The proof that $\bar{\mu}$ is a Gibbs state relative to $\overline{W}$ involves showing that

$$I = \int g(X)\exp[-\overline{W}(A|X)]\,\bar{\mu}_\Lambda(A, dX) \tag{A.17}$$

is independent of A for $A \subset \Lambda$. The first step is to note that

$$\int \bar{f}(Y)\,\bar{\mu}(dY) = \int f(Y)\,\mu(dY) \tag{A.18}$$

if f and $\bar{f}$ are related by (4.20), an equation which can be verified for continuous f using the definitions in (4.20) and (4.21) along with (3.11), or more generally by noting that (4.19) maps a subset $\mathscr{W}$ of $\mathscr{K}$ onto a subset $\overline{\mathscr{W}}$ in a manner such that $\mu(\mathscr{W})$ and $\bar{\mu}(\overline{\mathscr{W}})$ are identical.

We employ (A.18) for $\bar{f}$ defined by

$$\begin{aligned}\bar{f}(Y) &= g(Y\backslash\Lambda)\exp[-\overline{W}(A|Y\backslash\Lambda)] &&\text{if}\quad Y\cap\Lambda = A\\ &= 0 &&\text{if}\quad Y\cap\Lambda \neq A\end{aligned} \tag{A.19}$$

This makes the left side of (A.18) equal to (A.17). With the help of (4.20), the right side of (A.18) becomes

$$\int \bar{g}(X)\exp[-\overline{W}(A|X\,\Delta(R\backslash\Lambda))]\,\mu_\Lambda(A', dX) \tag{A.20}$$

where

$$\bar{g}(X) = g(X\,\Delta(R\backslash\Lambda)) \tag{A.21}$$

$$A' = A\,\Delta(R\cap\Lambda) \tag{A.22}$$

Next, one can show that, for $X \subset \mathscr{L}\backslash\Lambda$,

$$\overline{W}(A|X\,\Delta(R\backslash\Lambda)) = W(A'|X) - W(R\cap\Lambda|X) \tag{A.23}$$

Therefore

$$I = \int \bar{g}(X) e^{W(B \cap \Lambda | X)} e^{-W(A' | X)} \mu_\Lambda(A', dX) \tag{A.24}$$

which, by (4.15), is independent of A', and thus I, (A.17), is independent of A.

That the $\mu_\Lambda{}^\Omega$ of Theorem 4.8 satisfy (3.8) and (3.9) is a consequence of the definitions (4.26) and (4.25). Thus they define a probability measure in view of the fact that (4.24) implies that

$$\mu^\Omega(\mathscr{K}) = 1 \tag{A.25}$$

For A and B in Ω, (4.24) and (4.27) yield the equation

$$\nu_\Omega(A \cup B) = e^{W_\Omega(A|B)} \nu_\Omega(B) \tag{A.26}$$

This together with (4.26) and (4.27) may be used to check the validity of (4.15), using (3.11) to approximate the integrals.

Assume, on the other hand, that μ is some probability measure satisfying (4.14) with W replaced by W_Ω, and assume that $\Omega \subset \Lambda$. Then, as X (the second argument of W_Ω) must lie in $\mathscr{L} \backslash \Lambda$, it can, in view of (4.27), be replaced by $\varnothing$. Hence, upon integrating (4.15) with $g = 1$, we obtain

$$e^{-W(A|\varnothing)} \mu_\Lambda(A) = \mu_\Lambda(\varnothing) \tag{A.27}$$

and thus

$$\mu_\Lambda(A' \cup A'') = \nu_\Omega(A') \mu_\Lambda(\varnothing) / \nu_\Omega(\varnothing) \tag{A.28}$$

where A' and A'' stand for $A \cap \Omega$ and $A \cap (\Lambda \backslash \Omega)$, respectively. If (A.24) is summed over A' in Ω and A'' in $\Lambda \backslash \Omega$, we obtain

$$1 = 2^{|\Lambda \backslash \Omega|} \mu_\Lambda(\varnothing) / \nu_\Lambda(\varnothing) \tag{A.29}$$

which when substituted in (A.28) yields the right-hand side of (4.26). Of course, if μ_Λ and $\mu_\Lambda{}^\Omega$ are equal when $\Omega \subset \Lambda$, they are equal for all Λ because of the consistency condition (3.9).

APPENDIX B. DERIVATION OF LEMMA 5.6

Unfortunately, there are combinatorial errors in the section of Ref. 13 containing the result stated in Lemma 5.6. For the most part, these errors have been rectified by Del Grosso,[26] but for completeness we provide further details. In particular, we list the relevant definitions taking proper account of configurations with multiplicities.

The definition (5.11) of the lattice-gas correlation functions is extended to configurations with multiplicities by setting

$$\rho(X) = \begin{cases} \rho(\tilde{X}), & X = \tilde{X} \\ 0, & X \neq \tilde{X} \end{cases} \tag{B.1}$$

The truncated correlation functions $\rho^T(X)$ are then given by

$$\rho^T(\varnothing) = 0$$
$$\rho(X) = 1(X) + \sum_{n=1}^{\infty} \frac{X!}{n!} \sum_{\substack{(X_1, X_2, \dots, X_n) \\ \sum_r X_r = X}} \prod_{r=1}^{n} \frac{\rho^T(X_r)}{X_r!} \tag{B.2}$$

where the sum over configurations is over all decompositions of X into an *ordered* n-tuple $(X_1, X_2, \dots, X_n)$, and

$$1(X) = \begin{cases} 1, & X = \varnothing \\ 0, & X \neq \varnothing \end{cases} \tag{B.3}$$

The truncated correlation functions are related to the Ursell functions (5.45) by the equations

$$u(X) = \begin{cases} 2\rho^T(X) - 1 & \text{for } |X| = 1 \\ 2^{|X|}\rho^T(X) & \text{for } |X| > 1 \end{cases} \tag{B.4}$$

If we define a convolution,

$$\phi_1 * \phi_2(X) = \sum_{\substack{(X_1, X_2) \\ X_1 + X_2 = X}} \phi_1(X_1)\phi_2(X_2) \frac{X!}{X_1! X_2!} \tag{B.5}$$

we can write (B.2) as

$$\rho = \operatorname{Exp} \rho^T = \sum_{n=0}^{\infty} \frac{1}{n!} (\rho^T)^n \tag{B.6}$$

where $(\rho^T)^n = \rho^T * \rho^T * \cdots * \rho^T$ with n factors, and $(\rho^T)^0$ is the identity (B.3). Alternatively, we can write

$$\rho^T = \operatorname{Log} \rho = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\rho - 1)^n \tag{B.7}$$

The inverse ϕ^{-1} is defined by the equations

$$(\phi^{-1} * \phi)(X) = 1(X) \tag{B.8}$$

Let χ_Λ be a characteristic function for a set $\Lambda \subset \mathscr{L}$,

$$\chi_\Lambda(X) = \begin{cases} 1 & \text{for } \tilde{X} \subset \Lambda \\ 0 & \text{for } \tilde{X} \not\subset \Lambda \end{cases} \tag{B.9}$$

Then a scalar product is defined by

$$\langle \chi_\Lambda, \phi \rangle = \sum_X \chi_\Lambda(X) \frac{\phi(X)}{X!} \tag{B.10}$$

with the property

$$\langle \chi_\Lambda, \phi_1 * \phi_2 \rangle = \langle \chi_\Lambda, \phi_1 \rangle \langle \chi_\Lambda, \phi_2 \rangle \tag{B.11}$$

In addition, a map D_X is defined by

$$(D_X\phi)(Y) = \phi(X + Y) \tag{B.12}$$

This map obeys the rules:

$$\begin{aligned} \frac{D_X(\phi_1 * \phi_2)}{X!} &= \sum_{\substack{(X_1,X_2)\\ X_1+X_2=X}} \frac{D_{X_1}\phi_1}{X_1!} * \frac{D_{X_2}\phi_2}{X_2!} \\ \frac{D_X(\operatorname{Exp}\phi)}{X!} &= \left[\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{(X_1,X_2,\ldots,X_n)\\ \sum_r X_r = X}} \frac{D_{X_1}\phi}{X_1!} * \frac{D_{X_2}\phi}{X_2!} * \cdots * \frac{D_{X_n}\phi}{X^n!}\right] * \operatorname{Exp}\phi \end{aligned} \tag{B.13}$$

If we define the Boltzmann factor ϕ_Φ by

$$\phi_\Phi(X) = \begin{cases} 1 & X = \varnothing \\ e^{U(X)}, & X = \tilde{X} \\ 0 & X \neq \tilde{X} \end{cases} \tag{B.14}$$

where the energies $U(X)$ are given by (4.3), then in the low-activity region, $C(\Phi) < 1$, the techniques of Ref. 13 give

$$\rho(X) = \langle \chi_{\mathscr{L}}, \phi_\Phi^{-1} * D_X\phi_\Phi \rangle, \qquad \rho^T(X) = \langle \chi_{\mathscr{L}}, D_X\phi_\Phi{}^T/X! \rangle \tag{B.15}$$

where $\phi_\Phi{}^T$ is the Ursell–Mayer function given by

$$\phi_\Phi{}^T = \operatorname{Log}\phi_\Phi \tag{B.16}$$

The proof of the required cluster property now proceeds:

$$\begin{aligned} \sum_{\substack{X:i\in\tilde{X}\\ |X|=k}} |\rho^T(X)| &= \sum_{\substack{X:i\in\tilde{X}\\ |X|=k}} \left| \left\langle \chi_{\mathscr{L}}, \frac{D_X\phi_\Phi{}^T}{X!} \right\rangle \right| \\ &= \sum_{\substack{X:i\in\tilde{X}\\ |X|=k}} \left| \sum_Y \frac{\phi_\Phi{}^T(X+Y)}{X!\,Y!} \right| \\ &\leqslant \sum_{n=0}^{\infty} \sum_{T:|T|=k-1} \sum_{Y:|Y|=n} \frac{|D_i\phi_\Phi{}^T(T+Y)|}{(T+Y)!} \cdot \frac{(T+Y)!}{T!\,Y!} \\ &= \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \sum_{S:|S|=n+k-1} \frac{|D_i\phi_\Phi{}^T(S)|}{S!} \\ &\leqslant \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} |z| C(\Phi)^{n+k-1} \\ &= \frac{|z| C(\Phi)^{k-1}}{[1 - C(\Phi)]^k} \end{aligned} \tag{B.17}$$

In the last inequality we used the fact that

$$D_i\phi_\Phi{}^T(S) = D_i \operatorname{Log}\phi_\Phi(S) = \phi_\Phi^{-1} * D_i\phi_\Phi(S) \tag{B.18}$$

(which follows because D_i is a derivation) and the nontrivial result[26]

$$\sum_{S:|S|=r} \frac{|\phi_\Phi^{-1} * D_i\phi_\Phi(S)|}{S!} \leqslant |z|C(\Phi)^r \tag{B.19}$$

The equivalence of the cluster property (B.17) to (5.47) of Lemma 5.6 follows from (B.4).

APPENDIX C. THE ANALOG OF (5.8) WHEN p BECOMES INFINITE

Whereas the technique discussed below works for the $p = +\infty$ limit of any of the transformations discussed in this paper, the basic idea is most easily understood for decimation transformations and Niemeijer–van Leeuwen transformations for cells with an odd number of sites. For these transformations a particular assignment of τ values means that only a certain class of configurations of the object system, which we assume to be a finite set Ω, are permitted. The allowed configurations of such a modified object Hamiltonian receive, however, a Boltzmann weight determined by $H(\sigma)$ with no counter terms. Hence the same reasoning which resulted in (5.8) yields, in the present case, a formula

$$\exp[W_\Omega'(A'|B')] = \sum_{C\in\mathscr{C}(A'\cup B')} \exp U(C) \Big/ \sum_{C\in\mathscr{C}(B')} \exp U(C) \tag{C.1}$$

where A' and B' are subsets of the image system Ω', and shall remain fixed throughout the following discussion, and $\mathscr{C}(C')$ is the set of object-system configurations permitted if τ_i is $+1$ for all i in C' and -1 for all other i in Ω'.

Let D be those sites in Ω whose spin values are constrained by the values of τ_i for i in A'. Then there are classes $\mathscr{D}_+$ and $\mathscr{D}_-$ of subsets of D and a class $\mathscr{E}$ of subsets of $\Omega\backslash D$ such that C is in $\mathscr{C}(B')$ if and only if

$$C \cap D \in \mathscr{D}_- \tag{C.2}$$

and

$$C \cap (\Omega\backslash D) \in \mathscr{E} \tag{C.3}$$

whereas C is in $\mathscr{C}(A' \cup B')$ if and only if

$$C \cap D \in \mathscr{D}_+ \tag{C.4}$$

is satisfied together with (C.3). We may, therefore, reexpress the right side of (C.1) as

$$\left|\sum_{E\in\mathscr{E}} \sum_{D\in\mathscr{D}_+} e^{U(D\cup E)}\right| \Big/ \left|\sum_{E\in\mathscr{E}} \sum_{D\in\mathscr{D}_-} e^{U(D\cup E)}\right| \tag{C.5}$$

Next write

$$\sum_{D\in\mathscr{D}_+} e^{U(D\cup E)} = e^{U(E)} \sum_{D\in\mathscr{D}_+} e^{W(D|E)}$$

$$= e^{U(E)}e^{\theta(E)} \sum_{D\in\mathscr{D}_-} e^{W(D|E)} = e^{\theta(E)} \sum_{D\in\mathscr{D}_-} e^{U(D\cup E)} \tag{C.6}$$

where

$$e^{\theta(E)} = \Big[\sum_{\mathscr{D}\in D_+} e^{W(D|E)}\Big]\Big/\Big[\sum_{D\in\mathscr{D}_-} e^{W(D|E)}\Big] \tag{C.7}$$

Note that if the interactions are of finite range, $\theta(E)$ only depends on the portion of E that is near D. In particular, for (2.11), θ only depends on $E \cap D'$, where D' denotes all sites that are nearest neighbors of sites in D but not themselves members of D. On the other hand, if the interactions are not of finite range but W is continuous, (C.7) defines a function θ which is continuous on $\mathscr{K}$.

The combination of (C.1), (C.5), and (C.6) permits us to write

$$\exp W_\Omega'(A'|B') = \langle \exp \bar{\theta} \rangle_{\Omega,B'} \tag{C.8}$$

where the average on the right side is over the configurations C in $\mathscr{C}(B')$ with the weights appearing in the denominator of (C.1), and

$$\bar{\theta}(C) = \theta(C \cap (\Omega \backslash D)) \tag{C.9}$$

depends only on the properties of the configurations *outside* the set D.

In order to produce peculiarities in W', we of course take the formal limit of (C.8) as Ω and Ω' become infinite, with

$$B' = \Omega' \cap X'$$

for some (in general infinite) X'. The existence of a limit is no more (or less) problematical than it is in the case of (5.8) when the arguments of Section 5 do not apply.

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